1. Introduction: Pre-Requisite Knowledge - Tutorials

Give the following fluid and physical properties (at 20 Celsius and standard pressure) with a 4-digit accuracy.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air density</td>
<td></td>
<td></td>
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<tr>
<td>Water density</td>
<td></td>
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<tr>
<td>Air dynamic viscosity</td>
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<tr>
<td>Water dynamic viscosity</td>
<td></td>
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<tr>
<td>Gravity constant (in Brisbane)</td>
<td></td>
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<tr>
<td>Surface tension (air and water)</td>
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</tbody>
</table>

What is the definition of an ideal fluid? What is the dynamic viscosity of an ideal fluid?

From what fundamental equation does the Navier-Stokes equation derive: a- continuity, b- momentum equation, c- energy equation? Does the Navier-Stokes equation apply for any type of flow? If not for what type of flow does it apply for?

Sketch the streamlines of the following two-dimensional flow situations:

A- A laminar flow past a circular cylinder,

B- A turbulent flow past a circular cylinder,

In each case, show the possible extent of the wake (if any). Indicate clearly in which regions the ideal fluid flow assumptions are valid, and in which areas they are not.
2. Ideal Fluid Flow - Irrotational Flows - Tutorials

2.1 Quiz
- What is the definition of the velocity potential?
- Is the velocity potential a scalar or a vector?
- Units of the velocity potential?
- What is definition of the stream function? Is it a scalar or a vector? Units of the stream function?

For an ideal fluid with irrotational flow motion:
- Write the condition of irrotationality as a function of the velocity potential.
- Does the velocity potential exist for 1- an irrotational flow and 2- for a real fluid?
- Write the continuity equation as a function of the velocity potential.

Further, answer the following questions:
- What is a stagnation point?
- For a two-dimensional flow, write the stream function conditions.
- How are the streamlines at the stagnation point?

2.2 Basic applications
(1) Considering the following velocity field:

\[ V_x = y \times z \times t \]
\[ V_y = z \times x \times t \]
\[ V_z = x \times y \times t \]

- Is the flow a possible flow of an incompressible fluid?
- Is the motion irrotational? If yes: what is the velocity potential?

(2) Considering the following velocity field:

\[ V_x = 2 \times x \]
\[ V_y = -2 \times y \]

Is the motion irrotational? In the affirmative, what is the velocity potential?

(3) Draw the streamline pattern of the following stream functions:

(3.1) \( \psi = 50 \times x \)
(3.2) \( \psi = -20 \times y \)
(3.3) \( \psi = -40 \times x - 30 \times y \)
(3.4) \( \psi = -10 \times x^2 \) (from \( x = 0 \) to \( x = 5 \))

2.3 Two-dimensional flow
Considering a two-dimensional flow, find the velocity potential and the stream function for a two-dimensional flow having the following velocity components:
\[
V_x = -\frac{2 * x * y}{(x^2 + y^2)^2} \\
V_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}
\]

2.4 Applications

(a) Using the software 2DFlowPlus, investigate the flow field of a vortex (at origin, strength 2) superposed to a sink (at origin, strength 1). Visualise the streamlines, the contour of equal velocity ad the contour of constant pressure.

Repeat the same process for a vortex (at origin, strength 2) superposed to a sink (at x=-5, y=0, strength 1).

How would you describe the flow region surrounding the vortex.

(b) Investigate the superposition of a source (at origin, strength 1) and an uniform velocity field (horizontal direction, \(V = 1\)). How many stagnation point do you observe ? What is the pressure at the stagnation point ? What is the "half-Rankine" body thickness at x = +1 ? (You may do the calculations directly or use 2DFlowPlus to solve the flow field.)

(c) Using 2DFlowPlus, investigate the flow past a circular building (for an ideal fluid with irrotational flow motion). How many stagnation points is there ? Compare the resulting flow pattern with real-fluid flow pattern behind a circular bluff body (search Reference text in the library).

(d) Investigate the seepage flow to a sink (well) located close to a lake. What flow pattern would you use ?

Note : the software 2DFlowPlus is described in the lecture notes, Appendix D.

Reference


2.5 Basic equations (2)

Considering an two-dimensional irrotational flow of ideal fluid, which basic principle(s) is(are) used to determine the pressure field ?

Exercise Solutions

Exercise 2.2

Solution (1)

The equations satisfy the equation of continuity for :
\[ \frac{\partial V_x}{\partial x} = \frac{\partial V_y}{\partial y} = \frac{\partial V_z}{\partial z} = 0 \]

so that:
\[ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \]

The components of the vorticity are:
\[ \left( \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) = \frac{1}{2} * (x * t - x * t) = 0 \]
\[ \left( \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) = \frac{1}{2} * (y * t - y * t) = 0 \]
\[ \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \frac{1}{2} * (z * t - z * t) = 0 \]

hence vorticity is zero and the field could represent irrotational flow. The velocity potential would then be the solution of:
\[ \frac{\partial \phi}{\partial x} = -V_x = -y * z * t \phi = -x * y * z * t + f_1(y,z,t) \]
\[ \frac{\partial \phi}{\partial y} = -V_y = -x * z * t \phi = -x * y * z * t + f_2(x,z,t) \]
\[ \frac{\partial \phi}{\partial z} = -V_z = -x * y * t \phi = -x * y * z * t + f_3(x,y,t) \]

and hence:
\[ \phi = -x * y * z * t + f(t) \]

is a possible velocity potential.

**Solution (3)**
Irrotational flow motion.

**Solution (3)**
(1) Vertical uniform flow : \( V_o = -50 \text{ m/s} \)
(2) Horizontal uniform flow : \( V_o = -20 \text{ m/s} \)
(3) Uniform flow : \( V_o = 50 \text{ m/s} \) and \( \alpha = 127 \text{ degrees} \)
(4) Non uniform vertical flow

**Exercise 2.3**

**Solution**
In polar coordinate the velocity components are:
\[ V_x = \frac{-2 * \cos \theta * \sin \theta}{r^2} \quad V_y = \frac{\cos^2 \theta - \sin^2 \theta}{r^2} \]

Using:
\[ V_r = V_x \cos \theta + V_y \sin \theta \]
\[ V_\theta = V_y \cos \theta - V_x \sin \theta \]

we deduce:
\[ V_r = -\frac{\sin \theta}{r^2} \quad V_\theta = \frac{\cos \theta}{r^2} \]

In polar coordinates the velocity potential and stream function are:
\[ V_r = -\frac{\partial \phi}{\partial r} \quad V_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} \]
\[ V_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad V_\theta = \frac{\partial \psi}{\partial r} \]

Hence:
\[ \phi = -\frac{\sin \theta}{r} + \text{constant} \]

Discussion
The resulting flow pattern is a doublet at the origin aligned along the vertical axis. See CHANSON (2003), pp. 4-5- to 4-6.

Exercise 2.4
The software 2DFlow+ is installed on the engineering network.

Exercise 2.5
Solution
Bernoulli principle. See CHANSON (2003), pp. 2-8 to 2-9, p. 3-11.
3. Two-Dimensional Flows (1) Basic equations and flow analogies - Tutorials

3.1 Flow net
For a two-dimensional seepage under a impervious structure with a cutoff wall (Fig. E3-1), the boundary conditions are: \( H = 6 \text{ m}, K = 2.0 \text{ m/day} \).

(a) What is the hydraulic conductivity in m/s? What type of soil is it?
(b) Using the flow net, estimate the seepage flow parameter width of dam.

![Fig. E3-1 - Flow net beneath an impervious dam](image)

3.2 Flow net
For a two-dimensional seepage under sheetpiling with a permeable foundation (Fig. E3-2), the boundary conditions are: \( a = 9.4 \text{ m}, b = 4.7 \text{ m}, c = 4 \text{ m}, H = 2.5 \text{ m}, K = 2.0 \times 10^{-3} \text{ cm/s} \).

Using the flow net (with 5 stream tubes), estimate the seepage flow parameter width of dam.
3.3 Flow net
For a two-dimensional seepage under an impervious dam with apron (Lecture Notes, CHANSON 2003, Fig. 3-1B), the boundary conditions are: \( H = 80 \text{ m}, \ K = 5 \times 10^{-5} \text{ m/s} \).

(A) Calculate the seepage flow rate in presence of an apron.

(B) In absence of the cutoff wall, sketch the flow net and determine the seepage flow. Determine the pressure distribution along the base of the dam and beneath the apron. Calculate the uplift forces on the dam foundation and on the apron.

3.4 Flow net
For a two-dimensional seepage under an impervious dam with a cutoff wall (Fig. E3-3), the boundary conditions are: \( H_1 = 60 \text{ m}, H_2 = 5 \text{ m}, a = 60 \text{ m}, b = 100 \text{ m}, L = 70 \text{ m}, K = 1 \times 10^{-5} \text{ m/s} \).

(A) In absence of cutoff wall (i.e. \( a = 100 \text{ m} \)), sketch the flow net; determine the seepage flow; determine the pressure distribution along the base of the dam; calculate the uplift force.

(B) With the cutoff wall (i.e. \( a = 60 \text{ m} \)): same questions: sketch the flow net; determine the seepage flow; determine the pressure distribution along the base of the dam; calculate the uplift force

(C) Comparison and discuss the results.

*Use graph paper.*
Exercise Solutions

Exercise 3.1
Solution
q = 4.6 m²/day

Remark
See Lecture Notes (CHANSON 2003), pp. 3-3 to 3-14.

Exercise 3.2
Solution
q = 2.7 m²/day

Discussion
The flow pattern may be analysed analytically using a finite line source (for the sheet pile) and the theory of images. The resulting streamlines are the equipotentials of the sheet-pile flow.

Remember : A velocity potential can be found for each stream function. If the stream function satisfies the Laplace equation the velocity potential also satisfies it. Hence the velocity potential may be considered as stream function for another flow case. The velocity potential \( \phi \) and the stream function \( \psi \) are called "conjugate functions" (Chapter 2, paragraph 4.2).
Exercise 3.3

Solution

(A) The problem is similar to the flow net sketched in Figure 3-1B (Lecture Notes, Chapter 3, paragraph 2.2).

(B) In absence of cutoff wall, the streamlines are shorter and the seepage flow rate is greater. The uplift pressure on the dam foundation and apron become very significant. The apron structure would be subjected to high risks of uplift and damage.
4. Two-Dimensional Flows (2) Basic flow patterns - Tutorials

4.1 Doublet in uniform flow (1)
Select the strength of doublet needed to portray an uniform flow of ideal fluid with a 20 m/s velocity around a cylinder of radius 2 m.

4.2 Source and sink
A source discharging 0.72 m²/s is located at (-1, 0) and a sink of twice the strength is located at (+2, 0). For dynamic pressure at the origin of 7.2 kPa, $\rho = 1240$ kg/m³, find the velocity and dynamic pressure at (0, 1) and (1, 1).

4.3 Flow pattern (2)
In two-dimensional flow we now consider a source, a sink and an uniform stream. For the pattern resulting from the combinations of a source (located at (-L, 0)) and sink (located at (+L, 0)) of equal strength Q in uniform flow (velocity $+V_0$ parallel to the x-axis):
(a) Sketch streamlines and equipotential lines;
(b) Give the velocity potential and the stream function.
This flow pattern is called the flow past a Rankine body. W.J.M. RANKINE (1820-1872) was a Scottish engineer and physicist who developed the theory of sources and sinks. The shape of the body may be altered by varying the distance between source and sink (i.e. $2L$) or by varying the strength of the source and sink. Other shapes may be obtained by the introduction of additional sources and sinks and RANKINE developed ship contours in this way.
(c) What is the profile of the Rankine body (i.e. find the streamline that defines the shape of the body)?
(d) What is the length and height of the body?
(e) Explain how the flow past a cylinder can be regarded as a Rankine body. Give the radius of the cylinder as a function of the Rankine body parameter.

4.4 Flow pattern (3)
In two-dimensional flow we consider again a source, a sink and an uniform stream. But, the source is located at (+L, 0) and the sink is located at (-L, 0) (i.e. opposite to a Rankine body flow pattern). They are of equal strength q in an uniform flow (velocity $+V_0$ parallel to the x-axis).
Derive the relationship between the discharge q, the length L and the flow velocity such that no flow injected at the source becomes trapped into the sink.

Exercise Solutions

Exercise 4.1
Solution
(a) A doublet and uniform flow is analog to the flow past a cylinder of radius:
\[ R = \sqrt{-\frac{\mu}{V_0}} \]

where \( \mu \) is the strength of the doublet. Hence:

\[ \mu = -V_0 \cdot R^2 = 80 \text{ m}^3/\text{s} \]

Remark
See Lecture Notes (CHANSON 2003), pp. 4-11- to 4-17.

Exercise 4.3

Solution

The flow past a Rankine body is the pattern resulting from the combinations of a source and sink of equal strength in uniform flow (velocity \( +V_0 \) parallel to the x-axis):

\[
\phi = -V_0 \cdot r \cdot \cos \theta - \left( +\frac{q}{2 \cdot \pi} \cdot \ln \left( \frac{r_1}{r_2} \right) \right) \psi = -V_0 \cdot r \cdot \sin \theta - \left( +\frac{q}{2 \cdot \pi} \cdot (\theta_1 - \theta_2) \right)
\]

where the subscript 1 refers to the source, the subscript 2 to the sink and \( q \) is positive for the source located at \((-L, 0)\) and the sink located at \((+L, 0)\).

The profile of the Rankine body is the streamline \( \psi = 0 \):

\[
\psi = -V_0 \cdot r \cdot \sin \theta + \frac{q}{2 \cdot \pi} \cdot (\theta_1 - \theta_2) = 0
\]

\[
r = \frac{q \cdot (\theta_1 - \theta_2)}{2 \cdot \pi \cdot V_0 \cdot \sin \theta}
\]

The length of the body equals the distance between the stagnation points where:

\[
V = V_0 + \frac{q}{2 \cdot \pi \cdot r_1} - \frac{q}{2 \cdot \pi \cdot r_2} = V_0 + \frac{q}{2 \cdot \pi} \cdot \left( \frac{1}{r_s - L} - \frac{1}{r_s + L} \right) = 0
\]

and hence:

\[
L_{\text{body}} = 2 \cdot r_s = 2 \cdot L \cdot \sqrt{1 + \frac{q}{\pi \cdot L \cdot V_0}}
\]

The half-width of the body \( h \) is deduced from the profile equation at the point \((h, \pi/2)\):

\[
h = \frac{q \cdot (\theta_1 - \theta_2)}{2 \cdot \pi \cdot V_0}
\]

where: \( \theta_1 = \alpha \) and \( \theta_2 = \pi - \alpha \) and hence:

\[
\alpha = \frac{\pi}{2} - \frac{\pi \cdot h \cdot V_0}{q}
\]

But also:

\[
\tan \alpha = \frac{h}{L}
\]

So the half-width of the body is the solution of the equation:

\[
h = L \cdot \cot \left( \frac{\pi \cdot V_0}{q} \cdot h \right)
\]
Remark
See Lecture Notes (CHANSON 2003), pp. 4-24- to 4-28.

Exercise 4.4
See Lecture Notes (CHANSON 2003), pp. 4-28- to 4-31.
Flow net analysis & Graphical solutions - Discussion

A streamline is the line drawn so that the velocity vector is always tangential to it (i.e. no flow across a streamline). Some important characteristics of streamlines are:

1. There can be no flow across a streamline.
2. Streamlines converging in the direction of the flow indicate a fluid acceleration.
3. Streamlines do not cross.
4. In steady flow the pattern of streamlines does not change with time.
5. Solid stationary boundaries are streamlines provided that separation of the flow from the boundary does not occur.

The following theorems are proved as a consequence of kinetic energy considerations and are limited to irrotational flows of ideal fluid:

1. Irrotational motion is impossible if all of the boundaries are fixed.
2. Irrotational motion of a fluid will cease when the boundaries come at rest.
3. The pattern of irrotational flow which satisfies the Laplace equation and prescribed boundary conditions is unique and is determined by the motion of the boundaries.
4. Irrotational motion of a fluid at rest at infinity is impossible if the interior boundaries are at rest.
5. Irrotational motion of a fluid at rest at infinity is unique and determined by the motion of the interior solid boundaries.

Two-dimensional flow patterns can be represented visually by drawing a family of streamlines and the corresponding family of equipotential lines with the constants varying in arithmetical progression. The resulting network of lines is called a flow net. It is usual to select the $\phi$-lines and $\psi$-lines so that: $\delta \psi = \delta \phi = \delta c$. The flow net consists of an orthogonal grid that reduces to perfect squares, when the grid size approaches zero. In uniform flow region the squares are of equal size. In diverging flow their size increases, and in converging flow they decrease in size, in the direction of flow.

Basic characteristics of flow nets are:

1. Flow nets are based upon the assumption of irrotational flow of ideal fluid, but it is not necessary that the flow is steady.
2. For given boundary conditions there is only one possible flow pattern. This is a property of the Laplace equation.
3. Streamlines are everywhere tangent to the velocity vector. Fixed boundaries are streamlines. The volumetric flow rate between two streamlines is: $\delta q = \delta \psi$ where $\delta \psi$ is the difference in stream function value between adjacent streamlines.
4. There is no velocity component tangent to an equipotential line and hence the velocity vector must be everywhere normal to an equipotential line. Equipotential lines intersect fixed boundaries normally.
5. Streamlines and equipotential lines are orthogonal (i.e. intersect at right angles).
The problem of finding the flow net to satisfy given fixed boundaries is purely a graphical exercise: i.e., the construction of an orthogonal system of lines that compose boundaries and reduce to perfect squares in the limit as the number of lines increases. Typically the construction of a flow net includes:

1. Construction of the streamlines in the regions where the velocity distributions are evident: e.g., parallel or radial flow, fixed boundaries.
2. Sketch the remaining portions of streamlines with smooth curves.
3. Construction of the equipotential lines: (3a) normal to all streamlines and fixed boundaries, and (3b) forming squares with the streamlines. This is a process of trial-and-error. The diagonal of the 'squares' should form smooth curves. At stagnation points the occurrence of five-sided 'square' results from the impossibility to obtain infinite spacing of streamlines in the region.
4. Location of free surfaces by trial.
5. Tendency to separate is indicated where the velocity at a boundary reaches a maximum and thereafter decreases in the direction of flow.