Air Bubble Entrainment in Turbulent Water Jets Discharging into the Atmosphere[1]

H Chanson
ME, ENSHMG, INSTN, PhD (Cant.), EurIng., MIEAust., MIAHR
Senior Lecturer, Fluid Mechanics, Hydraulics and Environmental Engineering
Department of Civil Engineering, The University of Queensland, Brisbane QLD 4072, Australia

SUMMARY *High-velocity water jets discharging into air are characterised by free-surface aeration along the jet interfaces. The air entrainment is analysed in terms of air bubble diffusion. New analytical solutions of the void fraction profiles are developed for circular jets and two-dimensional jets. The equations are successfully compared with experimental data for both jet configurations with velocities between 4.5 and 36.5 m/s. For two-dimensional free-shear layers, the results imply air bubble turbulent diffusivity of the same order of magnitude as the eddy viscosity.*

NOTATION

The following symbols are used in this paper :

- C air concentration defined as the volume of air per unit volume of air and water; also called the void fraction;
- D_t turbulent diffusivity (m^2/s);
- D'' dimensionless diffusivity;
- d_o initial flow depth (m) or jet thickness;
- Fr Froude number;
- g gravity constant (m/s^2);
- J_o Bessel function of the first kind of order zero;
- J_1 Bessel function of the first kind of order one;
- K empirical constant;
- q discharge per unit width (m^2/s);
- Re Reynolds number; for circular jet : $Re = V_o{*}r_o/\nu_w$; for plane free-shear layer : $Re = V_o{*}d_o/\nu_w$;
- r radial direction (m) normal to the jet centreline;
- r_o nozzle radius (m);
- r_{90} distance (m) from the jet centreline where $C = 0.9$;
- t_r ramp height (m);
- t_s offset height (m);
- u independent variable;
- V velocity (m/s);
- x longitudinal flow direction (m);
- y direction (m) normal to the flow direction;
- y_{50} location (m) where $V = V_o/2$;
- α channel slope;
- α_n positive root of : $J_o(r_{90}{*}\alpha_n) = 0$;
- ϕ ramp angle;
- μ dynamic viscosity ($N.s/m^2$);
- ν kinematic viscosity (m^2/s) : $\nu = \mu/\rho$;
- ν_T eddy viscosity (m^2/s);
- ρ density (kg/m^3);
- Ψ spread angle of the air bubble diffusion layer computed between $C = 0.1$ and $C = 0.9$;
- $\varnothing$ diameter (m);

Subscript

- o initial flow conditions at the jet deflector or nozzle;
- r radial component;
- x longitudinal flow direction component;
- y transverse flow direction component;
- w water flow.

1 INTRODUCTION

In hydraulic and coastal structures, air entrainment may occur naturally or artificially. With high-velocity water jets discharging into the atmosphere, air bubble entrainment occurs along the air-water interfaces as shown on figure 1. High velocity turbulent jets are often used in hydraulic structures to dissipate energy and to induce or enhance air entrainment. Some examples include jet flows downstream of a ski jump at the toe of a spillway, water jets issued from bottom outlets and flows above a bottom aeration device along a spillway. Other applications of high velocity water jets include Pelton turbines, fire-fighting equipment and high velocity jets used in the coal industry to assist cutting and drilling. Free-surface aeration along the jet interfaces reduces the jet momentum and hence the momentum transfer to turbines, and decreases the length of fire-fighting jets and the cutting potential of the jet. In such applications air entrainment must be avoided.

[1] Paper C/95022 submitted to IEAust 16/10/95. Paper accepted 7/6/96

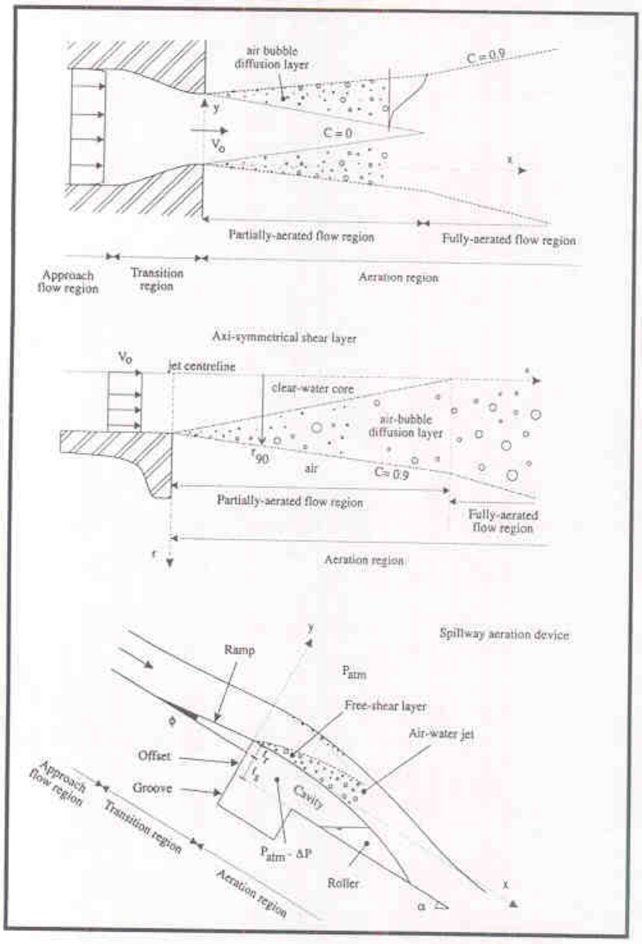

Figure 1 Sketch of turbulent water jets discharging into the atmoshpere; (a) two-dimensional jet; (b) circular jet; (c) spillway aeration device

The basic flow regions (treated in this paper) are the approach flow region, the transition region and the aeration region (fig. 1). The approach flow conditions characterise the flow immediately upstream of the deflector (e.g. nozzle convergence, ramp of spillway aerator). The transition region coincides with the length of the jump, the nozzle length or the length of the ramp. At the edge of the deflector or jet nozzle, there is a pressure change from a pipe flow pressure distribution or a quasi-hydrostatic pressure distribution, to a zero (or slightly negative) pressure gradient. From the edge of the nozzle up to the impact of the jet, the flow region is called the aeration flow region. At the beginning of the jet, there is a clear water core which is reduced in the flow direction (i.e. partially-aerated flow region). If the jet is long enough, a fully-aerated flow region starts developing downstream of the point where the central part of the jet becomes aerated. The rate of expansion of the jet differs between the partially-aerated and fully-aerated flow regions as shown on figure 1.

A number of researchers studied the properties of circular turbulent water jets : e.g., Hoyt and Tailor[9] (fire-fighting equipment), Dodu[6] (Pelton turbine) and Van de Sande and Smith[17] (mixing devices). Most studies used photographic techniques. Some studies (e.g. Heraud[8] , Ervine and Falvey[7]) discussed the effects of air entrainment on jet spreading and jet breakup of circular water jets. Kawakami[10] showed that flow aeration downstream of a ski jump reduced the jet length. His work was based upon field data. Another type of application is the aeration devices on spillways (fig. 1(C)). Aerators are designed to deflect high velocity flow away from the chute surface and to aerate the flow. Three studies (Shi et al.[15], Low[11], Chanson[2]) performed air concentration measurements above aerators in the free-shear layer. One of these (Chanson[2]) recorded also the velocity distributions.

In this paper, the air bubble diffusion at the jet interfaces is studied analytically. A new solution of the diffusion equation for circular jets is proposed. The results are compared with two-dimensional jet calculations and with experimental results. Details of the experiments are summarised in table I. Note that the work applies primarily to the partially-aerated flow region of high-velocity water jets.

2 AIR BUBBLE DIFFUSION

2.1 Fundamental equations

High-velocity water jets discharging into the atmosphere are characterised by air bubble entrainment along the jet interfaces. The advective diffusion of air bubbles is governed by the continuity equation for air :

$$\operatorname{div}(C\,\vec{V}) = \operatorname{div}(D_t * \overrightarrow{\operatorname{grad}}\, C) \qquad (1)$$

where C is the air content (i.e. void fraction), V is the velocity and D_t is the turbulent diffusivity. The diffusion equation can be solved for both two-dimensional and circular jets.

For a circular water jet discharging into the atmosphere (fig. 1(B)), the continuity equation for air becomes :

$$\frac{V_x}{D_r} * \frac{\partial C}{\partial x} + \frac{C}{D_r} * \frac{\partial V_x}{\partial x} = \frac{1}{r} * \frac{\partial}{\partial r}\left(r * \frac{\partial C}{\partial r} \right) \qquad (2)$$

where x is the longitudinal direction, r is the radial direction, V_x is the velocity component in the x-direction and D_r is the turbulent diffusivity in the radial direction. Equation (2) implies a constant diffusivity in the radial direction and is applied to a small control volume along a streamtube.

Separating the variables, the air content :

$$C = u * \exp\left(- \frac{D_t}{V_x} * \alpha_n{}^2 * x \right) \qquad (3)$$

is a solution of the continuity equation provided that u is a function of r only satisfying Bessel's equation of order zero. D_t averages the effects of the transverse turbulent diffusion and of the longitudinal velocity gradient. Away from the nozzle edge (i.e. x >> 0), $D_t = D_r$. At each position x, the diffusivity D_t is assumed independent of the transverse location r. The solution of equation (2) is a series of Bessel functions :

$$C = 0.9 - \frac{1.8}{r_{90}} * \sum_{n=1}^{+\infty} \frac{J_0(r * \alpha_n)}{\alpha_n * J_1(r_{90} * \alpha_n)} * \exp\left(-\frac{D_t}{V_o} * \alpha_n^2 * x\right)$$

Circular jet (4)

where V_o is the flow velocity at the nozzle, r_{90} is the radial distance from the centreline where C = 0.9, J_o is the Bessel function of the first kind of order zero, α_n is the positive root of : $J_o(r_{90}*\alpha_n) = 0$, and J_1 is the Bessel function of the first kind of order one (see appendix A for full details). Equation (4) is valid in both the partially-aerated and fully-aerated flow regions.

For two-dimensional water jets, the analytical solution of the continuity equation for air (eq. (1)) in the partially-aerated flow region yields :

$$C = \frac{1}{2} * \left(1 - \operatorname{erf}\left(\frac{y}{2 * \sqrt{\frac{D_t}{V_o} * x}}\right)\right)$$

Two-dimensional jet (5)

where x and y are defined on figure 1(A). The function erf is defined as :

$$\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} * \int_0^u \exp(-t^2) * dt \qquad (6)$$

Equation (5) was first applied to air bubble entrainment in free-shear flows by Chanson[2,3] assuming $\partial C/\partial x << \partial C/\partial y$ and for a small control volume $\{\partial x, \partial y\}$ limited between two streamlines. In equation (5), D_t averages the effect of the turbulence and of the longitudinal velocity gradient. Note that equation (5) is valid only in the partially-aerated flow region. Formulations of equations (4) and (5) for spatially-dependent diffusion coefficients are developed in appendix B.

2.2 Comparison with experimental data

Some researchers recorded the air concentration distributions at the air-water interfaces of water jets discharging into the atmosphere (table I). The scope of these studies covered circular jets and water jets issued from a deflector down an inclined channel (i.e. spillway aeration device, fig. 1(C)).

The re-analysis of the data indicates that both the air concentration profiles and the air-water velocity distributions are smooth, continuous and differentiable (Heraud [8], Chanson [2]). The air-water flow behaves as a homogeneous mixture for C < 0.90. It was therefore decided to investigate the air bubble diffusion for 0 and 90% of air contents.

The solutions of the diffusion equation (eq. (4) and (5)) have been compared successfully with all the data. Some examples are shown on figures 2 and 3. For circular jets, figure 2 presents the theoretical air content distributions (eq. (4)) computed for several values of the dimensionless diffusivity D" defined as :

$$D" = \frac{D_t * x}{V_o * r_{90}^2} \qquad (7)$$

Equation (4) is compared with experimental data. The experimental flow conditions are detailed in table II. The experiment reference number is given in column 2. On figure 2, the data are presented for several values of x/d_o and the corresponding value of D" is deduced from the best data fit. Figure 3 compares equation (5) with experimental data obtained at the lower interface of a spillway aerator flow (i.e. as on fig. 1(C)). Note that, for a spillway aeration device, the upper free-surface of the nappe is not a free-shear layer (see discussion in CHANSON 1989). And this case is not covered in this paper.

Figures 2 and 3 show a good agreement between data and theory. Equations (4) and (5) were obtained assuming an uniform velocity distribution and constant diffusivity (D_r or D_t). Although these assumptions are not realistic in turbulent shear layer flows, the close agreement between data and analytical solutions suggests that the air bubble diffusion process is little affected by the turbulent shear layer.

3 AIR-WATER FLOW CHARACTERISTICS

3.1 Momentum shear layer

There is few information on the air-water velocity distribution in the jet shear layers. The only published set of data (Chanson[2]) included two series of velocity measurements with two-dimensional water jets. The analysis of the results (Chanson[4]) indicated that the velocity distribution followed Goertler's solution of the momentum equation for a plane shear layer :

$$\frac{V}{V_o} = \frac{1}{2} * \left(1 + \operatorname{erf}\left(\frac{K * (y - y_{50})}{x}\right)\right) \qquad (8)$$

where K is a constant estimated as $K = 17.7^2$ and y_{50} is the location where $V = V_o/2$. For monophase shear layers Rajaratnam[12] and Schlichting[14] obtained K = 11.0 and 13.5 respectively.

3.2 Turbulent diffusivity

For several experiments the author calculated the turbulent diffusivity D_t satisfying equations (4) and (5).

---

^{2}Chanson[4] stated K = 25. But a re-analysis indicates in fact that $K = 25/\sqrt{2}$.

Table I

Experimental flow configurations for air concentration and velocity measurements in turbulent water jets discharging into the atmosphere

Ref. (1)	Slope α (deg.) (2)	Deflector geometry (3)	Nb of Exp. (4)	V_o (m/s) (5)	d_o(m) (6)	Fr_o (7)	P_N (8)	Comments (9)
Two-dimensional jet								
Shi et.al.[15]	49.00	No offset. Ramp: t_r=0.015 m, ϕ = 5.7 deg	1	14.0	0.058	18.6	1.0	Aerator model
Low[11]	51.30	Offset: t_s = 0.03m Ramp: t_r = 0.03m, ϕ = 5.7 deg.	5	4.2 to 9.45	0.050	6-13.5	0-0.6	Aerator model (W = 0.25m). Partially-developed inflow
Chanson[2]	52.33	Offset: t_s = 0.03m No ramp	2	9.24	0.023	19.5	0.01 to 0.5	Aerator model (W = 0.25m). Partially-developed inflow
			12	6.15 to 11.3	0.035	10.5-19.5	0 to 1.6	
			2	5.32	0.081	6.0	0.07 to 0.3	
Circular jet					$\varnothing_o$			
Heraud[8]	0	Smooth and rough nozzles	6	11.6 to 36.5	0.033	20.4 to 64.2	0	Horizontal water jets. Smooth and roung injectors. Smooth and rough turbulent inflow conditions
Ervine and Falvey[7]	0	Jet nozzle		3.3 to 29.6	$\varnothing_o$ = 0.05, 0.1	20.4 & 64.1	0	Horizontal water jets
Ruff et al.[13]	90	Nozzle	3	38.5 & 56.4	$\varnothing_o$ = 0.0191 & 0.0095	88.9 & 184.7	0	Vertical water jets discharging downwards. Uniform flow injector (13.6:1 contraction) and fully-developed flow injector
Tseng et al.[16]	90	Nozzle	2	49.1	$\varnothing_o$ = 0.0095	160.8	0	Vertical water jets discharging downwards. Nozzle arrangement as Ruff et al. [13]

Notes:

Nb of Exp	number of experiments
d_o, V_o:	initial flow depth (or nozzle diameter) and flow velocity
Fr_o:	range of Froude numbers
P_N:	range of pressure gradient numbers: $PN = \Delta P/(\rho_w * g * d_o)$ (for aeration devices only)
t_r,t_s,ϕ:	ramp height, offset height, ramp angle (see fig. 1(C))

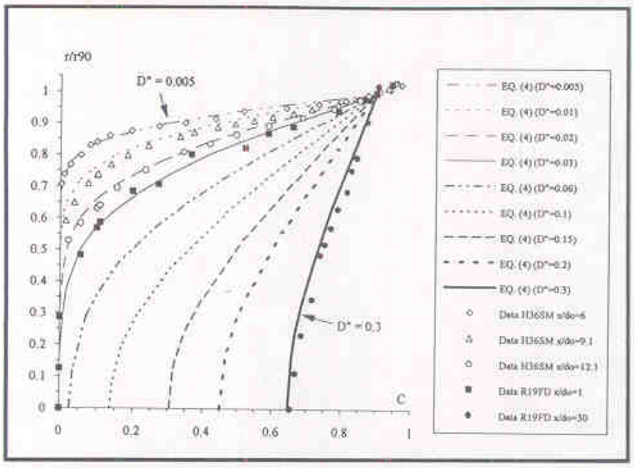

Figure 2 Air concentration distribution at the free-surface of circular water jets discharging into the atmosphere — comparison between equation (4) and experimental data (HERAUD 1966, run H36SM, RUFF et al., run R19FD)

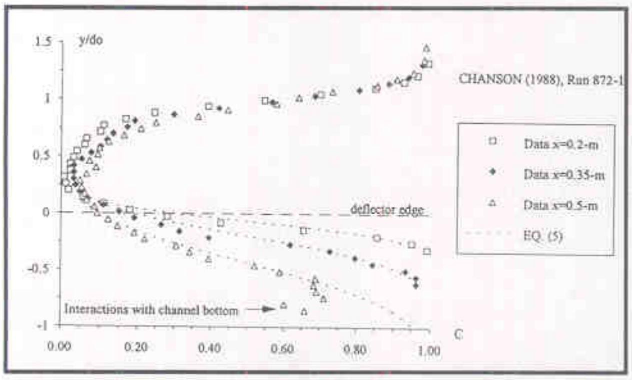

Figure 3 Air concentration distribution at the air-water interface of a two-dimensional free-shear layer flow discharging into the atmosphere — comparison between equation (5) and experimental data (CHANSON 1988)

The results are reported in table II and plotted on figure 4 as a function of the Reynolds number : for circular jet $Re = V_o * r_o / \nu_w$, for two-dimensional jet $Re = V_o * d_o / \nu_w$. For circular jets the diffusivity D_t is deduced from the appropriate value of D" (eq. (7)) which best fits the data. For two-dimensional jet experiments, the turbulent diffusivity at the free-shear layer air-water interface D_t is estimated as :

$$D_t = \frac{1}{2} * \frac{V_o * x}{1.2817} * (\tan\Psi)^2 \qquad (9)$$

where Ψ is the observed initial spread angle measured between C = 10% and C = 90%.

First the wide scatter of the results is obvious.

Secondly, the turbulent diffusivity for Heraud's[8] experiments is much smaller than other results. For the experiments of HERAUD, the jet nozzle was extremely smooth and the inflow was quasi-uniform. With rougher deflectors, free-surface instabilities develop, enhancing the aeration process and the air bubble diffusion coefficient.

Third, Ruff et al.[13] and Tseng et al.[16] investigated the effects of inflow conditions (i.e. uniform inflow and fully-developed inflow). Their results (fig. 4 and 6, table II) show clearly larger diffusion coefficient for fully-developed inflow conditions. The initial flow mixing enhances the diffusion process and the diffusivity values.

Figure 5 presents the ratio D_t/ν_T where ν_T is the turbulent kinematic viscosity. D_t/ν_T describes the combined effect of : 1- the difference in the diffusion of a discrete particle (e.g. air bubble) and the diffusion of a small coherent fluid structure, and 2- the influence of the particles on the turbulence field (e.g. turbulence damping). For plane shear layers, ν_T can be deduced from equation (8) as Goertler's solution of the momentum equation implies a constant eddy viscosity across the shear layer :

$$\nu_T = \frac{1}{4 * K^2} * x * V_o \qquad (10)$$

The results (fig. 5) suggest that the turbulent diffusivity D_t (taking into account both the transverse diffusion and the longitudinal fluid acceleration) and the eddy viscosity are of the same order of magnitude for plane shear layers. A similar order of magnitude is observed also with sediment-laden open channel flows. But no information on air bubble diffusion was previously available.

No air-water velocity measurement is available for circular water jets and the analysis cannot be extended to the axi-symmetrical jet configuration.

3.3 Spread angle

The spread angle Ψ of air bubble diffusion layer provides some simple and practical information on the rate of diffusion of air bubbles. For two-dimensional jet experiments, the spread angle Ψ (computed between C = 0.1 and C = 0.9) may be estimated as :

$$\Psi = 0.698 * V_o^{0.630}$$

{Data : SHI et al. 1983, LOW 1986, CHANSON 1988} (11a)

where Ψ is in degrees and V_o is in m/s. Equation (11a) is an empirical correlation obtained from experiments detailed in table II. With circular jets with quasi-uniform inflow, the spread angle is independent of the nozzle diameter and initial flow turbulence. It may be estimated as :

$$\Psi = 0.02884 * V_o$$

{Data : Heraud[8], Ruff et al.[13], Tseng et al. 16} (11b)

For circular water jets with fully-developed inflow, the spread angle is about 4.5 degrees.

Table II
Experimental values of the turbulent diffisivity at the air-water interface of turbulent water jets

Ref. (1)	Run (2)	V_o m/s (3)	d_o m (4)	ψ degrees (5)	D_t m^2/s (6)	Remarks (7)
Bi-dimensional free-shear layers Chanson[2]	860-1	10.71	0.0198	3.71	1.41E-3	Aerator model (W = 0.25m)
	860-2	10.71	0.0198	3.89	1.55E-3	
Chanson[2]	870-1	10.65	0.0324	3.05	2.17E-3	
	870-2	10.65	0.0324	2.47	1.42E-3	
	871-1	9.54	0.0318	2.89	1.50E-3	
	871-2	9.54	0.0318	2.40	1.04E-3	
	872-1	12.01	0.0330	3.24	2.81E-3	
	872-2	12.01	0.0330	2.53	1.71E-3	
	873-1	8.72	0.0313	2.31	7.29E-4	
	873-2	8.72	0.0313	2.42	8.04E-4	
	874-1	7.00	0.0300	2.03	4.52E-4	
	874-2	7.00	0.0300	1.96	4.21E-4	
	1050	7.56	0.0352	2.32	7.56E-4	
	1051	10.56	0.0345	3.28	1.70E-3	
Chanson[2]	880-1	6.21	0.0691	1.90	3.58E-4	Aerator model (W = 0.25m)
	880-2	6.21	0.0691	1.84	3.36E-4	
Low[11]	A7	4.61	0.0465	1.95	5.58E-4	Aerator model (W = 0.25m)
	A8	5.19	0.0474	1.80	4.86E-4	
	A9	5.71	0.0529	2.60	9.99E-4	
	A10	6.64	0.0567	3.77	2.93E-3	
	A11	8.36	0.0559	2.89	1.63E-3	
Shi et al.[15]	Fig. 5	10.73	0.0608	4.98	1.00E-5	Aerator model
Circular jets			$\varnothing_o$			
Heraud[8]	H11SM	11.60	0.0330	0.36	2.72E-5	Smooth pipe and smooth injector
	H36SM	36.50	0.0330	1.02	3.52E-4	
Heraud[8]	H11RO	11.60	0.0330	0.39	3.42E-5	Rough pipe and smooth injector
	H36RO	36.50	0.0330	1.20	4.53E-4	
Ruff et al.[3]	R19FD	38.46	0.0191	4.71	6.44E-3	Fully-developed inflow
	R95FD	56.39	0.0095	3.99	7.09E-3	
Ruff et al.[13]	R95UN	56.39	0.0095	1.64	1.10E-3	Uniform inflow
Tseng et al.[16]	T95FD	49.10	0.0095	5.10	3.79E-3	Fully-developed inflow (*)
Tsent et al.[16]	T95UN	49.10	0.0095	1.30	4.79E-4	Uniform inflow (*)

Notes:

V_o flow velocity at the end of the deflector
d_o initial jet thickness or jet daimeter
ψ initial spread angle measured between C = 10% and C = 90%
(*) Surrounding air pressure: P_{atm} = 1 atm

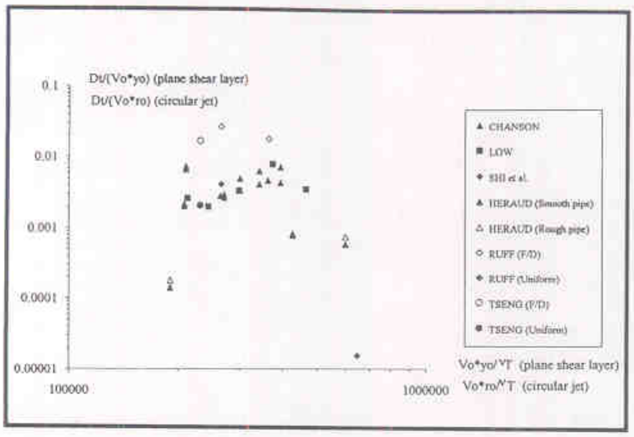

Figure 4 Dimensionless diffusivity as a function of the Reynolds number for water jets discharging into the atmosphere

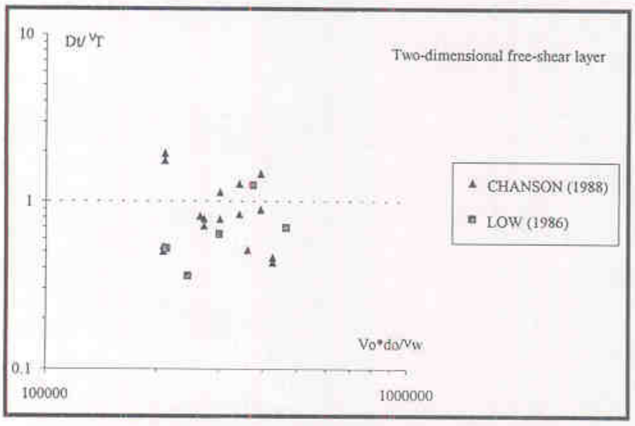

Figure 5 Ratio of the turbulent diffusivity over the eddy viscosity for two-dimensional free-shear layer flows discharging into the atmosphere

3.4 Effects of inflow conditions and nozzle roughness

3.4.1 Effects of the deflector/injector roughness

Several researchers attempted to investigate the effects of the roughness of the deflector (or jet nozzle) on the flow properties in the developing flow region.

Heraud[8] presented superb photographic results. He compared the jet characteristics between a smooth injector and injectors with three different roughnesses (k_s = 0.1, 0.2, 0.3 mm). He introduced also single roughness elements in the smooth injector. Using high-speed photographs, he observed that the injector roughness affects the jet only near the injector, that the effect decreases with the distance from the injector and becomes negligible for $x/Ø_o > 10$ where $Ø_o$ is the jet nozzle diameter. Individual localised roughness elements enhance the jet contraction because of the separation behind the roughness elements. HERAUD noted also a reduction of the jet dispersion with individual roughness elements.

3.4.2 Effects of inflow conditions

Ruff et al.[13] and Tseng et al.[16] investigated the effects of inflow conditions on the 'atomization' process. For identical flow conditions, they used both uniform flow conditions and fully-developed inflow conditions. A typical result is shown on figure 6 where the uniform flow data are in black symbols while the corresponding data with fully-developed inflow conditions are shown with while symbols. For identical x/d_o, figure 6 shows a faster spread of the jet for fully-developed inflow conditions.

The air diffusion process is enhanced considerably with fully-developed inflow conditions. The length of the clear-water core is shorter as air bubbles diffuse rapidly inside the jet. And the outer edge of the spray is wider.
Applications of these results imply that Pelton turbine jets and water cutting jet must be designed with uniform inflow conditions. This is obtained by designing a smooth nozzle shape with a large contraction ratio, installing a calming section upstream of the nozzle (e.g. with flow straighteners), and reducing the flow turbulence in the water supply system.

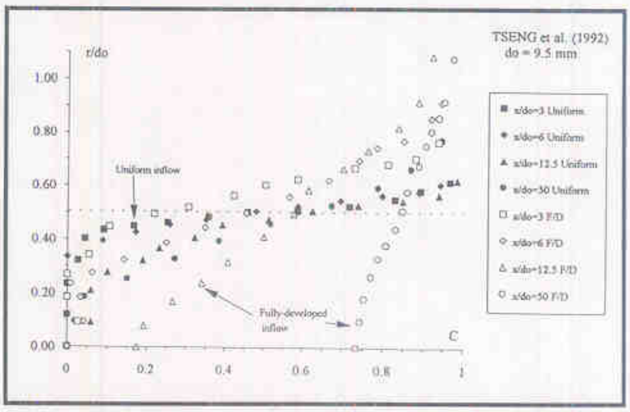

Figure 6 Effect of the inflow conditions on the free-surface aeration process (Data: TSENG et al. 1992) V_o = 49.1 m/s, d_o = 0.0095 m, F/D: fully-developed inflow, Uniform: uniform inflow conditions

4 CONCLUSION

In high-velocity water jets discharging into the atmosphere, free-surface aeration is observed along the jet interfaces. The distributions of air concentration can be approximated by a simple advective diffusion theory (eq. (4) and (5)) for air contents between 0 and 90% and mean velocities from 4.6 to 36.5 m/s. Although the analytical equation is obtained assuming an uniform velocity distribution, the close agreement between data and theory suggests that the shear flow has little effect on the turbulent diffusion process. Additional information is provided on the turbulent diffusivity for circular and plane jet experiments with velocities up to 36.5 m/s. In plane free-shear layer, the turbulent diffusivity is of the same order of magnitude as the turbulent kinematic viscosity.

The accurate prediction of the air-water flow properties will enable better prediction of the quantity of air entrained and the momentum transfer between the air-water jet and the surrounding gas. It must be emphasised that the analysis is based upon a small number of data. Additional air content and air-water velocity measurements are required to get a better understanding of the interactions between air bubble diffusion and turbulent field characteristics.

5 ACKNOWLEDGMENTS

The author acknowledges the support of the Department of Hydraulics and Ocean Engineering, National Cheng Kung University (Taiwan R.O.C.) and Professor LEE J.F. (NCKU). He thanks the anonymous reviewers for their helpful comments.

6 REFERENCES

1. Carslaw HS and Jaeger JC. Conduction of Heat in Solids. London:Oxford University Press, 2nd ed, 1959.

2. Chanson H. A Study of Air Entrainment and Aeration Devices on a Spillway Model. New Zealand: Dept. of Civil Engrg, University of Canterbury, 1988. Ph.D. thesis, Ref. 88-8.

3. Chanson H. Study of Air Entrainment and Aeration Devices. Jl of Hyd Res, IAHR 1989;27(3):301-19.

4. Chanson H. (1993). Velocity Measurements within High Velocity Air-Water Jets. Jl of Hyd Res, IAHR 1993;31(3):365-82 & (6):858.

5. Crank J. The Mathematics of Diffusion. London: Oxford University Press, 1956.

6. Dodu J. Etude de la Couche Limite d'Air autour d'un Jet d'Eau à Grande Vitesse. ('Study of the Boundary Layer around a High Velocity Water Jet). Proc 7th IAHR Congress, Lisbon, Portugal, paper D6 (in French), 1957.

7. Ervine DA and Falvey HT. Behaviour of Turbulent Water Jets in the Atmosphere and in Plunge Pools. Proc Instn Civ Engrs, UK, Part 2, Mar. 1987:83, 295-314. Discussion : Part 2, Mar.-June 1988: 85, 359-63, 1987.

8. Heraud D. Dispersion des Jets Liquides; Influence des Rugosités de Paroi, ('Dispersion of Liquid Jets; Influence of the Wall Roughness.') France: University Grenoble 1, 1966. PhD Thesis.

9. Hoyt JW and Taylor JJ. Turbulence Structure in a Water Jet Discharging in Air. Physics of Fluids Pt. II 1977;20(10):S253-57.

10. Kawakami K. A Study of the Computation of Horizontal Distance of Jet Issued from Ski-Jump Spillway. Proc. JSCE 1973;219(11):37-44 (in Japanese).

11. Low HS. Model Studies of Clyde Dam Spillway aerators. Research Report No. 86-6, Dept. of Civil Eng., Univ. of Canterbury, Christchurch, New Zealand, 1986.

12. Rajaratnam N. Turbulent Jets. Elsevier Scientific, Development in Water Science, 5, New York, USA, 1976.

13. Ruff GA, Sagar AD and Faeth GM. Structure and Mixing Properties of Pressure-Atomized Sprays. AIAA Jl 1989;27(7):901-8.

14. Schlichting H. Boundary Layer Theory. New York, McGraw-Hill, 7th edition, 1979.

15. Shi Q, Pan S, Shao Y and Yuan X. Experimental Investigation of Flow Aeration to prevent Cavitation Erosion by a Deflector. Shuili Xuebao (Jl of Hydraulic Engrg) Beijing, China, 1983; 3:1-13 (in Chinese).

16. Tseng LK, Ruff GA and Faeth GM. (1992). Effects of Gas Density on the Structure of Liquid Jets in Still Gases. AIAA Jl 1992;30(6):1537-44.

17. Van de Sande E and Smith JM. Surface Entrainment of Air by High Velocity Water Jets. Chem. Eng. Science 1973;28:1161-68.

APPENDIX A
Air bubble diffusion in circular water jets

Considering a circular water jet discharging into the atmosphere, the differential form of the continuity equation for air is :

$$\frac{\partial}{\partial x}(C * V_x) = \frac{1}{r} * \frac{\partial}{\partial r}\left(D_r * r * \frac{\partial C}{\partial r}\right) \quad \text{(A-1)}$$

where x is the longitudinal direction, r is the radial direction, V_x is the velocity component in the x-direction and D_r is the turbulent diffusivity in the radial direction. It is assumed that $\partial C/\partial x << \partial C/\partial r$.

Assuming [H1] a steady flow, [H2] a constant diffusivity D_r (in the radial direction) and [H3] for a small control volume delimited by streamlines (i.e. streamtube), equation (A-1) becomes :

$$\frac{V_x}{D_r} * \frac{\partial C}{\partial x} + \frac{C}{D_r} * \frac{\partial V_x}{\partial x} = \frac{1}{r} * \frac{\partial}{\partial r}\left(r * \frac{\partial C}{\partial r}\right) \quad \text{(A-2)}$$

Separating the variables, the air content :

$$C = u * \exp\left(-\frac{D_t}{V_x} * \alpha_n^2 * x\right) \quad \text{(A-3)}$$

is a solution of the continuity equation provided that u is a function of r only satisfying Bessel's equation of order zero :

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} * \frac{\partial u}{\partial r} + \alpha_n{}^2 * u = 0 \qquad \text{(A-4)}$$

The diffusivity term D_t averages the effects of the turbulent diffusion and the longitudinal velocity gradient in the left-hand side of equation (A-2). D_t is defined as :

$$D_t = D_r * \left(1 + \frac{1}{\alpha_n{}^2 * D_r} * \frac{\partial V_x}{\partial x}\right) \qquad \text{(A-5)}$$

Away from the deflector (i.e. x >> 0), the term $\partial V_x/\partial x$ becomes negligible and $D_t = D_r$. It is assumed that [H4] the diffusivity D_t is independent of the transverse location r at each position x.
For the set of boundary conditions :

$C = 0.9$ for $r = r_{90}$ and $x > 0$
$C = 0$ for $x < 0$

an analytical solution can be obtained. The solution of equation (A-2) is a series of Bessel functions :

$$C = 0.9 - \frac{1.8}{r_{90}} * \sum_{n=1}^{+\infty} \frac{J_o(r*\alpha_n)}{\alpha_n * J_1(r_{90}*\alpha_n)} * \exp\left(-\frac{D_t}{V_x} * \alpha_n{}^2 * x\right) \qquad \text{(A-6)}$$

where J_o is the Bessel function of the first kind of order zero[3], α_n is the positive root of : $J_o(r_{90}*\alpha_n) = 0$, and J_1 is the Bessel function of the first kind of order one[4].

Equation (A-6) was deduced from numerical computations by CARSLAW and JAEGER (1959) for several values of the dimensionless parameter $D" = (D_t*x)/(V_x*r_{90}{}^2)$. Typical computations are shown on figure 2.
Note that equation (A-6) is valid in both the partially-aerated flow region and the fully-developed flow region. Equation (A-6) is indeed a three-dimensional solution of the diffusion equation. Equation (a-6) implies that the jet becomes fully-aerated for D" > 0.03.

APPENDIX B
Spatially-dependent diffusion coefficients

If the turbulent diffusivity depends upon the longitudinal coordinate x (i.e. in the direction of the advection/diffusion process) but is independent of the other coordinates and time, the same type of solution as above (i.e. eq. (4) and (5)) can be used (CRANK 1959). In this case, the solutions of the diffusion equation are :

$$C = 0.9 - \frac{1.8}{r_{90}} * \sum_{n=1}^{+\infty} \frac{J_o(r*\alpha_n)}{\alpha_n * J_1(r_{90}*\alpha_n)} * \exp(-\alpha_n{}^2 * X) \qquad \text{Circular jet (4b)}$$

$$C = 1 - \operatorname{erf}\left(\frac{y}{\sqrt{2*X}}\right) \qquad \text{Two-dimensional jet (5b)}$$

where :

$$X = \int_0^x \frac{D_t}{V_x} * du$$

[3] $J_o(u) = 1 - \frac{u^2}{2^2} + \frac{u^4}{2^2 * 4^2} - \frac{u^6}{2^2 * 4^2 * 6^2} + \ldots$

[4] $J_1(u) = \frac{u}{2} - \frac{u^3}{2^2 * 4} + \frac{u^5}{2^2 * 4^2 * 6} - \frac{u^7}{2^2 * 4^2 * 6^2 * 8} + \ldots$

H CHANSON

Dr Chanson is a Senior Lecturer in Fluid Mechanics, Hydraulics and Environmental Engineering at the Department of Civil Engineering, The University of Queensland. His research interests include experimental investigations of multiphase flows and particularly air-water mixtures, design of hydraulic and coastal structures and modelling of water quality in rivers, estuaries and the ocean. He wrote a book on stepped spillways (Pergamon Publ. 1995) and over twenty-five international refereed journal papers.