$$\frac{\partial \delta_2}{\partial x} + \frac{1}{V_o} \times \frac{\partial V_o}{\partial x} \times (2 \times \delta_2 + \delta_1) = \frac{\tau_o}{\rho \times V_o^2}$$

where δ_1 and δ_2 are respectively the displacement thickness (Eq. (3-2)) and the momentum thickness (Eq. (3-3)).
6- The present development is based upon the elegant presentation of LIGGETT (1994, pp. 200-201).

The von Karman momentum integral equation may be used to solve a boundary layer problem by assuming a particular velocity distribution. This approach is valid for both laminar and turbulent boundary layers. It was first used by POHLHAUSEN (1921). In modern times, the momentum integral equation is rarely used for laminar flows, but it is commonly applied to turbulent boundary layer flows.

3.2 Application to a flat plate

Using the solution of the Blasius equation, the momentum integral equation may be applied to a laminar boundary layer. Let us assume that the velocity profile above a flat plate may be expressed as :

$$\frac{v_x}{V_o} = a_0 + a_1 \times \frac{y}{\delta} + a_2 \times \left(\frac{y}{\delta}\right)^2 \quad (3\text{-}28)$$

where a_0, a_1 and a_2 are undetermined coefficients, The coefficients are determined from the boundary conditions : $v_x(y=0) = 0$, $v_x(y=+\infty) = V_o$ and $(\partial v_x/\partial y) = 0$ for $y = +\infty$. The velocity distribution is found to be:

$$\frac{v_x}{V_o} = 2 \times \frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \quad (3\text{-}29)$$

The von Karman momentum equation for a flat plate becomes :

$$V_o^2 \times \frac{\partial}{\partial x}(\delta_2) = \frac{\tau_o}{\rho} \quad (3\text{-}30)$$

After the appropriate substitutions, the laminar boundary layer characteristics may be derived :

$$\delta = 5.48 \times \frac{x}{\sqrt{Re_x}} \quad (3\text{-}31)$$

$$\delta_1 = 1.83 \times \frac{x}{\sqrt{Re_x}} \quad (3\text{-}32)$$

$$\delta_2 = 0.73 \times \frac{x}{\sqrt{Re_x}} \quad (3\text{-}33)$$

The overall shear force on a plate of length L is :

$$\frac{\int_{x=0}^{L} \tau_o \times dx}{\frac{1}{2} \times \rho \times V_o^2 \times L} = \frac{1.46}{\sqrt{Re_L}} \quad (3\text{-}34)$$

The results are summarised in Table 3-2 and they are compared with the solution of the Blasius equation.
The above approach may be extended to a velocity profile which satisfies a polynomial of third degree (see Exercises). The results for a polynomial of fourth degree are listed in Table 3-2 in which they are compared with the parabolic velocity distribution results and the exact solution of the Blasius equation. The

comparison shows that the momentum integral results with a polynomial of fourth degree are close to the exact solution and even the results with a quadratic polynomial are within 10% of the theoretical solution which is reasonable.

Note

A polynomial of second degree is called a quadratic polynomial, or parabolic function.

Table 3-2 - Effects of the velocity distribution assumptions on the laminar boundary layer characteristics above a flat plate : comparison between the von Karman momentum integral equation and the Blasius equation

Boundary layer parameter	Approximate solutions		Theoretical solution
Velocity distribution:	$\frac{v_x}{V_o} = 2 \times \frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$	$\frac{v_x}{V_o} = \frac{y}{\delta} \times \left(2 - 2 \times \left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3\right)$	Blasius equation
$\delta =$	$5.48 \times \frac{x}{\sqrt{Re_x}}$	$5.84 \times \frac{x}{\sqrt{Re_x}}$	$4.91 \times \frac{x}{\sqrt{Re_x}}$
$\delta_1 =$	$1.83 \times \frac{x}{\sqrt{Re_x}}$	$1.75 \times \frac{x}{\sqrt{Re_x}}$	$1.72 \times \frac{x}{\sqrt{Re_x}}$
$\delta_2 =$	$0.73 \times \frac{x}{\sqrt{Re_x}}$	$0.685 \times \frac{x}{\sqrt{Re_x}}$	$0.664 \times \frac{x}{\sqrt{Re_x}}$
$\frac{\tau_o}{\frac{1}{2} \times \rho \times V_o^2} =$	$\frac{0.730}{\sqrt{Re_x}}$	$\frac{0.685}{\sqrt{Re_x}}$	$\frac{0.664}{\sqrt{Re_x}}$
$\frac{\int_{x=0}^{L} \tau_o \times dx}{\frac{1}{2} \times \rho \times V_o^2 \times L} =$	$\frac{1.46}{\sqrt{Re_L}}$	$\frac{1.370}{\sqrt{Re_L}}$	$\frac{1.328}{\sqrt{Re_L}}$

3.3 Discussion

This technique may be applied to other boundary layer flows including those with some longitudinal pressure gradient $\partial P/\partial x \neq 0$. POHLHAUSEN (1921) solved the momentum integral equation for a velocity distribution which satisfies a polynomial of fourth degree :

$$\frac{v_x}{V_o} = a_0 + a_1 \times \frac{y}{\delta} + a_2 \times \left(\frac{y}{\delta}\right)^2 + a_3 \times \left(\frac{y}{\delta}\right)^3 + a_4 \times \left(\frac{y}{\delta}\right)^4 \qquad (3\text{-}35)$$

where a_0 to a_4 are undetermined constants.

At the boundary (y = 0), the boundary conditions are $v_x(y=0) = 0$ and

$$\left(\frac{\partial^2 v_x}{\partial y^2}\right)_{y=0} = -\frac{\rho}{\mu} \times V_o \times \frac{\partial V_o}{\partial x} \qquad (3\text{-}36)$$

The above condition comes from the boundary layer equation (see below). Outside of the boundary layer ($y \geq \delta$), the velocity profile satisfies further : $v_x(y=\delta) = V_o$, and $(\partial v_x/\partial y) = (\partial^2 v_x/\partial y^2) = 0$ for $y = \delta$.

Note

Equation (3-36) is a simple rewriting of the boundary layer equation (Eq. (3-6)). At the boundary (y =0), Equation (3-6) becomes :

$$\left(-\frac{1}{\rho} \times \frac{\partial P}{\partial x} - g \times \frac{\partial z_o}{\partial x} + \frac{\mu}{\rho} \times \frac{\partial^2 v_x}{\partial y^2}\right)_{y=0} = 0$$

Using the differential form of the Bernoulli equation along the streamline in the ideal fluid flow region :

$$V_o \times \frac{\partial V_o}{\partial x} + g \times \frac{\partial z_o}{\partial x} + \frac{1}{\rho} \times \frac{\partial P}{\partial x} = 0$$

and replacing into Equation (3-6), it yields :

$$V_o \times \frac{\partial V_o}{\partial x} + \left(\frac{\mu}{\rho} \times \frac{\partial^2 v_x}{\partial y^2}\right)_{y=0} = 0$$

Introducing the dimensionless parameter :

$$\lambda = \frac{\rho}{\mu} \times \delta^2 \times \frac{\partial V_o}{\partial x} \qquad (3\text{-}37)$$

the parameter λ may be interpreted as a ratio of a longitudinal pressure gradient to some viscous force (LIGGETT 1994). Physically a positive value of λ indicates a favourable pressure gradient ($\partial P/\partial x < 0$) and a negative value denotes an adverse pressure gradient ($\partial P/\partial x > 0$). For a flat plate in absence of pressure gradient, $\lambda = 0$.

The parameters of Equation (3-35) are found to be :

$$a_0 = 0 \qquad (3\text{-}38a)$$

$$a_1 = 2 + \frac{\lambda}{6} \qquad (3\text{-}38b)$$

$$a_2 = -\frac{\lambda}{2} \qquad (3\text{-}38c)$$

$$a_3 = -\left(2 - \frac{\lambda}{2}\right) \qquad (3\text{-}38d)$$

$$a_4 = 1 - \frac{\lambda}{6} \qquad (3\text{-}38e)$$

If λ is a constant, the velocity profiles are self-similar. In a general case, the parameter λ varies with x and the parameters a_0 to a_4 vary with distance. That is, the shape of the velocity profile changes along the boundary layer.

The boundary layer characteristics may be expressed in terms of the dimensionless parameter λ :

$$\frac{\delta_1}{\delta} = \frac{3}{10} - \frac{\lambda}{120} \qquad (3\text{-}39)$$

$$\frac{\delta_2}{\delta} = \frac{37}{315} - \frac{\lambda}{945} + \frac{\lambda^2}{9072} \qquad (3\text{-}40)$$

The dimensionless boundary shear stress equals :