Minimum Specific Energy and Critical Flow Conditions in Open Channels

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Abstract: In open channels, the relationship between the specific energy and the flow depth exhibits a minimum, and the corresponding flow conditions are called critical flow conditions. Herein they are reanalyzed on the basis of the depth-averaged Bernoulli equation. At critical flow, there is only one possible flow depth, and a new analytical expression of that characteristic depth is developed for ideal-fluid flow situations with nonhydrostatic pressure distribution and nonuniform velocity distribution. The results are applied to relevant critical flow conditions: e.g., at the crest of a spillway. The finding may be applied to predict more accurately the discharge on weir and spillway crests.

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Introduction

Considering an open channel flow, the free surface is always at atmospheric pressure, the driving force of the fluid motion is gravity, and the fluid is incompressible and Newtonian. Newton's law of motion leads to the Navier–Stokes equations. The integration of the Navier–Stokes equations along a streamline, assuming that the fluid is frictionless, the volume force potential (i.e., gravity) is independent of the time, for a steady flow (i.e., $\partial V/\partial t=0$) and an incompressible flow (i.e., ρ=constant), yields

$$\frac{P}{\rho}+g*z+\frac{v^2}{2}=\text{constant} \tag{1}$$

where ρ=fluid density; g=gravity acceleration; z=elevation aligned along the vertical direction and positive upward); P=pressure; and V=velocity (Henderson 1966; Liggett 1993; Chanson 1999, 2004). Eq. (1) is the local form of the Bernoulli equation.

In this study, the singularity of the depth-averaged Bernoulli principle for open channel flow is detailed, i.e., the critical flow conditions. Detailed expressions of the critical flow properties are derived for the general case of nonhydrostatic and nonuniform velocity distributions. The results are then applied to the rating curve of weir crest acting as discharge meter.

Application to Open Channel Flows

In open channels, it is common to use the depth-averaged Bernoulli equation within the frame of relevant assumptions (e.g., Liggett 1993)

$$H=\frac{1}{d}*\int_0^d\left[\left(\frac{v(y)^2}{2*g}+z(y)+\frac{P(y)}{\rho*g}\right)\right]*dy$$
$$=\beta*\frac{V^2}{2*g}+\Lambda*d+z_0=\text{constant} \tag{2}$$

where H=depth-averaged total head; z_0=bottom elevation; d=flow depth; β=momentum correction coefficient (or Boussinesq coefficient); and V=depth-averaged velocity

$$V=\frac{1}{d}*\int_0^d v*dy \tag{3}$$

y=distance normal to the channel bed and Λ=pressure correction coefficient defined as

$$\Lambda=\frac{1}{2}+\frac{1}{d}*\int_0^d\frac{P(y)}{\rho*g*d}*dy \tag{4}$$

For a flat channel assuming a hydrostatic pressure distribution, the pressure correction coefficient Λ is unity and the depth-averaged total head H equals

$$H=\beta*\frac{V^2}{2*g}+d+z_0=E+z_0 \tag{5}$$

where E=depth-averaged specific energy.

Critical Flow Conditions

Considering a short smooth transition, and assuming a constant flow rate, the relationship between the specific energy ($E=H-z_0$) and flow depth exhibits a characteristic shape (e.g., Fig. 1). For a given cross-section shape, the specific energy is minimum for flow conditions (d_c, V_c) called the critical flow conditions. The concept of critical flow conditions was first developed by Bélanger (1828) as the location where $d=Q^2/(g*A^2)$ for a flat channel, where Q=flow rate and A=flow cross-section area. It was associated with the idea of minimum specific energy by Bakhmeteff (1912, 1932). Both Bélanger and Bakhmeteff devel-

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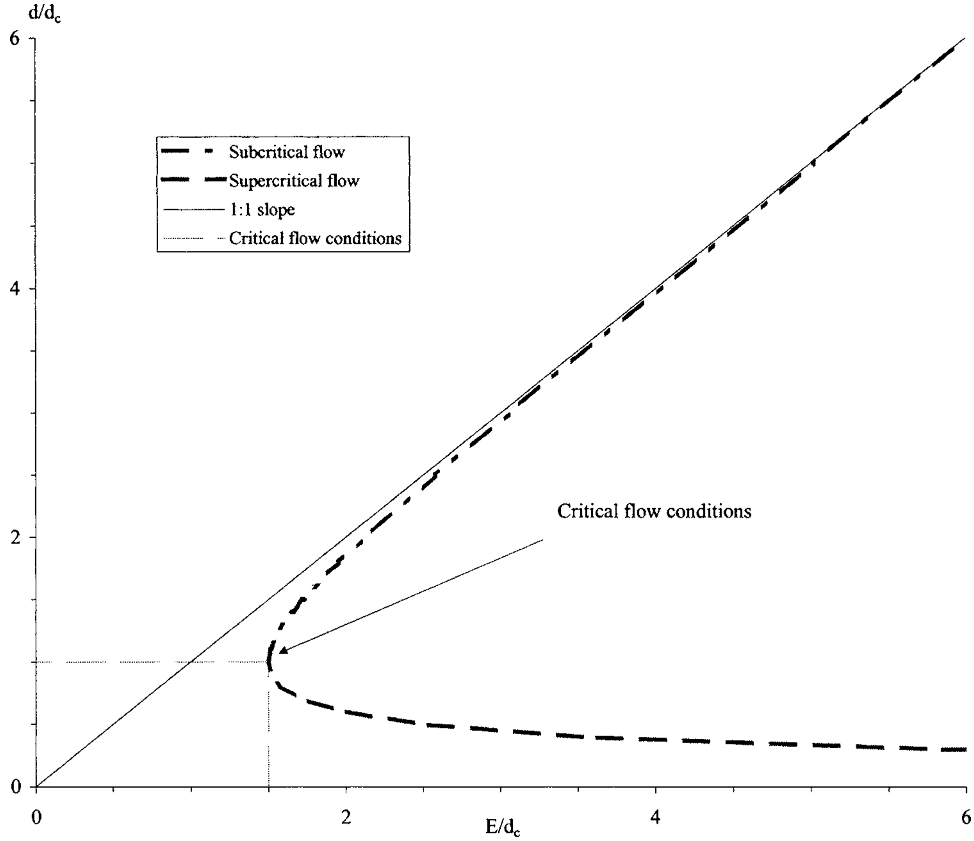

Fig. 1. Relationship between dimensionless specific energy E/d_c and dimensionless flow depth d/d_c in smooth rectangular channel assuming hydrostatic pressure distribution

oped the concept of critical flow in relation with the singularity of the backwater equation for $d=Q^2/(g*A^2)$ (i.e., critical flow conditions).

A typical situation with minimum specific energy is shown in Fig. 2 where critical flow conditions occur at the point of maximum invert elevation: i.e., the weir crest. Assuming a smooth frictionless overflow, the depth-averaged Bernoulli equation states

$$H=(z_0)_{\text{crest}}+E_{\min}=(z_0)_{\text{crest}}+\Lambda_{\text{crest}}*d_c+\beta_{\text{crest}}*\frac{V_c^2}{2*g} \quad (6a)$$

where H=upstream total head; $(z_0)_{\text{crest}}$=crest elevation; $E_{\min}$=minimum specific energy (Fig. 2); and β_{crest} and Λ_{crest}=, respectively, momentum and pressure correction coefficients at the crest. Note that, at the crest, the y direction is exactly vertical, the streamlines are curved, the pressure distribution is not hydrostatic, and the velocity distribution is not uniform.

For a rectangular channel, the continuity and Bernoulli equations give two equations in terms of the critical flow depth and depth-averaged velocity are as follows:

$$\text{Bernoulli equation } E_{\min}=\Lambda_{\text{crest}}*d_c+\beta_{\text{crest}}*\frac{V_c^2}{2*g} \quad (6b)$$

$$\text{continuity equation } q=V_c*d_c=C_D*\sqrt{g}*\left(\frac{2}{3}*E_{\min}\right)^{3/2} \quad (7)$$

where q=discharge per unit width and C_D=dimensionless discharge coefficient.

If the minimum specific energy $E_{\min}$ and flow rate per unit width q are known parameters, the combination of Eqs. (7) and (6b) gives a third-order polynomial equation in terms of the dimensionless flow depth at crest (i.e., $d_c/E_{\min}$)

$$\left(\frac{d_c}{E_{\min}}\right)^3-\left(\frac{d_c}{E_{\min}}\right)^2*\frac{1}{\Lambda_{\text{crest}}}+\frac{1}{2}*\frac{\beta_{\text{crest}}*C_D^2}{\Lambda_{\text{crest}}}*\left(\frac{2}{3}\right)^3=0 \quad (8)$$

Eq. (8) has one, two, or three real solutions depending upon the sign of the discriminant Δ (see the Appendix)

$$\Delta=\left(\frac{1}{3*\Lambda_{\text{crest}}}\right)^6*4*\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2*(\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2-1) \quad (9)$$

The solution of Eq. (8) gives an expression of the dimensionless critical depth as a function of the pressure correction coefficient, momentum correction coefficient, and discharge coefficient. Further the Bernoulli equation implies for an overflow

$$0\leqslant\Lambda_{\text{crest}}*\frac{d_c}{E_{\min}}\leqslant 1 \quad (10)$$

The detailed solutions of Eq. (8) are developed in the Appendix. Meaningful solutions exist only for $\Delta\leqslant 0$. These solutions are plotted in Fig. 3 as $d_c/E_{\min}*\Lambda_{\text{crest}}$ versus $\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2$. The analytical results are compared with the reanalysis of experimental data (Fawer 1937; Vo 1992), flow net analysis (Fawer 1937), and detailed analytical solution (Fawer 1937). The experimental flow conditions are listed in Table 1 and one complete data set is presented in Fig. 4.

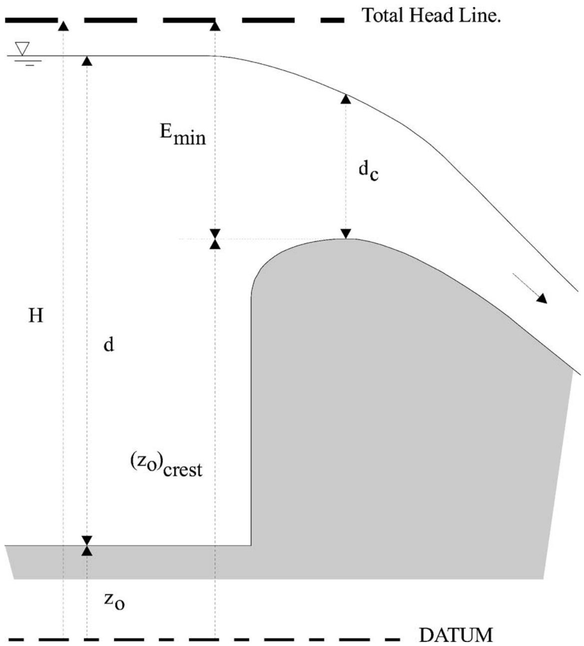

Fig. 2. Critical flow conditions at weir crest—definition sketch

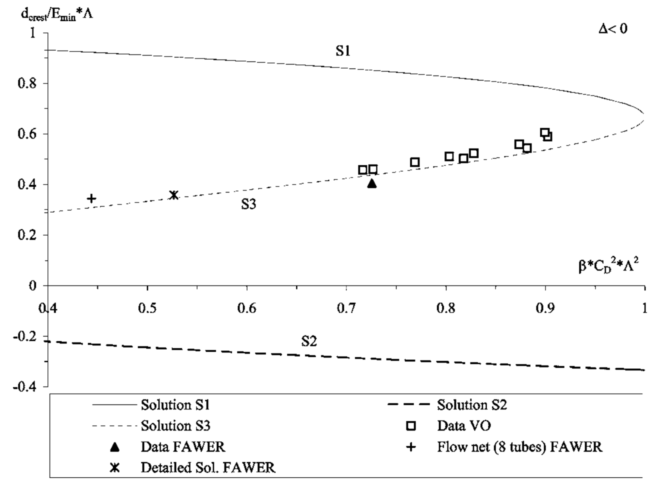

Fig. 3. Dimensionless critical flow depth at crest of weir: $d_c/E_{\min}*\Lambda_{\text{crest}}$ versus $\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2$—comparison with experimental data (Fawer 1937, Vo 1992), flow net analysis with eight stream tubes (Fawer 1937), and detailed analytical solution (Fawer 1937)

Overall the data (Fig. 3) follow closely the solution $S3$

$$\text{Solution } S3 \quad \frac{d_c}{E_{\min}} = \frac{2}{3*\Lambda_{\text{crest}}} * \frac{1}{2} * [1 - \cos(\delta/3) + \sqrt{3*\{1-[\cos(\delta/3)]^2\}}] \tag{11}$$

where

$$\cos\delta = 1 - 2*\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2 \tag{12}$$

It is unclear why experimental data do not follow the solution $S1$, although it is conceivable that $S1$ might be an unstable solution.

For a hydrostatic pressure distribution ($\Lambda_{\text{crest}}=1$) and a uniform velocity distribution ($\beta_{\text{crest}}=1$), the discharge coefficient is unity and the flow depth at crest equals

$$\frac{d_c}{E_{\min}} = \frac{2}{3} \tag{13}$$

for example, an ideal fluid flow above a broad-crested weir.

Discussion

The analysis of Eq. (8) yields basic conclusions. First the product $\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2$ must be less than or equal to unity

$$\beta_{\text{crest}}*C_D^2*\Lambda_{\text{crest}}^2 \leq 1 \tag{14}$$

As the momentum correction coefficient β_{crest} is equal to or larger than unity, Eq. (14) implies that discharge coefficients larger than unity may be obtained only when the crest pressure distribution is less than hydrostatic.

Second, for transition from sub- to supercritical flow, experimental results (Fig. 3) indicate that

$$\frac{d_c}{E_{\min}}*\Lambda_{\text{crest}} \leq \frac{2}{3} \tag{15}$$

That is, the dimensionless critical flow depth equals the solution $S3$ (Appendix), and that the streamline curvature usually implies ($\Lambda_{\text{crest}}<1$) at a weir crest. Therefore Eq. (15) does not imply $d_c/E_{\min}<2/3$. The result is well known for overflow circular weirs (e.g., Vo 1992; Chanson and Montes 1998).

Third the pressure and velocity distributions at the crest can be predicted using ideal fluid flow theory (i.e., potential flow theory). Hence β_{crest} and Λ_{crest} may be calculated theoretically because the entire flow field may be predicted assuming an ideal fluid with irrotational flow motion. Assuming a two-dimensional flow, the vertical distributions of pressure and velocity may be accurately determined numerically by a computational method, graphically by a flow net analysis, or analytically for simple geometries (e.g., Fawer 1937; Rouse 1946; Jaeger 1956; Vallentine 1969). Then the relevant parameters become $d_c/E_{\min}$ and C_D, or q, d_c, and $E_{\min}$.

Table 1. Summary of Reanalyzed Experimental Flow Measurements

Reference	Configuration	Measurements	Remarks
Fawer (1937)	Circular weir (R=0.0325 m) Vertical upstream wall 3:2 downstream slope Weir height: 0.0315 and 0.3325 m	Invert pressure distributions. Vertical distributions of pressure and velocity (Pitot tube).	2.5 m long 0.303 m wide upstream flume.
Vo (1992)	Circular weir (R=0.0095–0.1516 m) Upstream slope: 90, 75, 60° Downstream slope: 75, 60, 45°	Invert pressure distributions. Vertical distributions of pressure and velocity (LDV).	1.8 m long 0.254 m wide upstream flume. (Also Ramamurthy et al. 1992)

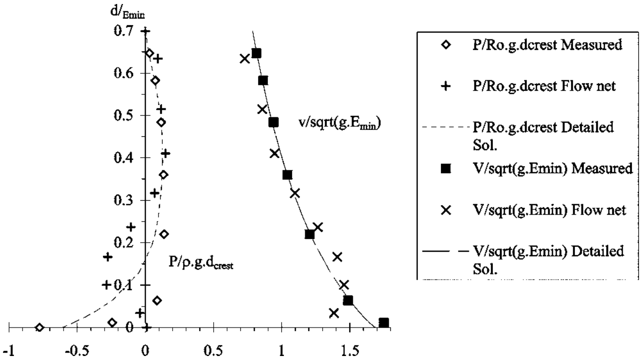

Fig. 4. Dimensionless pressure and velocity distributions at crest of weir: $P/\rho * g * d_c$ and $V/\sqrt{g * E_{\min}}$ versus $d_c/E_{\min}$ (Fawer 1937; d_c=0.0537 m, $E_{\min}$=0.0768 m, R=0.0325 m)—comparison among experimental data, flow net analysis (eight stream tubes), and detailed analytical solution (ideal fluid)

Application: Spillway Crest as Discharge Meter

If the momentum and pressure correction coefficients may be predicted theoretically, the continuity and Bernoulli equations imply that a spillway crest may be used as an accurate discharge meter using the solution of Eq. (8). In practice the upstream head above spillway crest (i.e., $E_{\min}$) is known and the unknown is the flow rate q. If the flow depth at the crest (i.e., the critical flow depth d_c) is measured, Eq. (11) and Fig. 3 provide the value of the discharge coefficient C_D satisfying Eq. (8), and the flow rate is deduced from Eq. (7). In contrast with the free overfall, a spillway crest is a better discharge meter because the correction coefficients and the discharge coefficient are close to unity.

Considering a circular weir (R=4.5 m) with an upstream vertical weir in a rectangular channel, the upstream head above crest is 10.6 m and the measured depth on the crest is 7.4 m. Compute the flow rate. Fawer (1937) derived the flow net solution of this case. His graphical result based upon eight stream tubes predicted: β_{crest}=1.0 and Λ_{crest}=0.5. For these values, Eq. (11) and Fig. 3 imply that C_D=1.44. Fawer conducted the corresponding experiment that yielded C_D=1.4, while his detailed potential flow solution gave C_D=1.38. Both results are close to the analytical prediction [Eq. (11) and Fig. 3].

Conclusion

Critical flow conditions in open channel are reanalyzed using the depth-averaged form of the Bernoulli equation. At critical flow, a new analytical expression of the critical flow depth is derived for ideal-fluid flows. The result is applied to critical flow situations with nonhydrostatic pressure distribution and nonuniform velocity distribution. It yields pertinent information on the flow properties at a weir crest. The findings may be applied to predict accurately the discharge at the crest at spillways and weirs, by combining Eq. (11) and Fig. 3 with simple ideal fluid flow theory (e.g., flow net analysis).

In real-fluid flows, boundary friction induces a flow region affected by shear and momentum exchange: i.e., a developing boundary layer. Converging and accelerating flow situations (e.g., spillway intake) have generally thin boundary layers. Present ideal fluid results may be applied to short transitions of real fluids to a satisfactory degree of approximation, but they are not applicable to long waterway: e.g., undular flow in a culvert barrel.

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Appendix: Critical Flow Depth

The continuity and Bernoulli equations give an expression for the critical flow depth as the solution of

$$\left(\frac{d_c}{E_{\min}}\right)^3 - \left(\frac{d_c}{E_{\min}}\right)^2 * \frac{1}{\Lambda_{\text{crest}}} + \frac{1}{2} * \frac{\beta_{\text{crest}} * C_D^2}{\Lambda_{\text{crest}}} * \left(\frac{2}{3}\right)^3 = 0 \tag{16}$$

Eq. (16) has one, two, or three real solutions depending upon the sign of the discriminant Δ

$$\Delta = \frac{1}{\Lambda_{\text{crest}}^6} * \frac{4}{3^6} * \beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2 * (\beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2 - 1) \tag{17}$$

For $\Delta > 0$, Eq. (16) has only one real solution

$$\frac{d_c}{E_{\min}} * \Lambda_{\text{crest}} = \sqrt[3]{\frac{1}{27} * (1 - 2 * \beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2) + \sqrt{\Lambda_{\text{crest}}^6 * \Delta}} + \sqrt[3]{\frac{1}{27} * (1 - 2 * \beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2) - \sqrt{\Lambda_{\text{crest}}^6 * \Delta}} + \frac{1}{3} \tag{18}$$

For Δ=0, the following condition holds:

$$\beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2 - 1 = 0 \tag{19}$$

The only physical solution of Eq. (16) is

$$\frac{d_c}{E_{\min}} * \Lambda_{\text{crest}} = \frac{2}{3} = \frac{2}{3} * \sqrt{\beta_{\text{crest}}} * C_D * \Lambda_{\text{crest}} \tag{20}$$

For Δ=0, the second real solution is negative: $d_c/E_{\min} = -1/(3 * \Lambda_{\text{crest}})$.

For $\Delta < 0$ there are three real solutions

$$\text{solution } S1 \left(\frac{d_c}{E_{\min}} * \Lambda_{\text{crest}}\right)_1 = \frac{2}{3} * \left[\frac{1}{2} + \cos(\delta/3)\right] \tag{21a}$$

$$\text{solution } S2 \left(\frac{d_c}{E_{\min}} * \Lambda_{\text{crest}}\right)_2 = \frac{2}{3} * \frac{1 - \cos(\delta/3) - \sqrt{3 * \{1 - [\cos(\delta/3)]^2\}}}{2} \tag{21b}$$

$$\text{solution } S3\left(\frac{d_c}{E_{\min}} * \Lambda_{\text{crest}}\right)_3 = \frac{2}{3} * \frac{1 - \cos(\delta/3) + \sqrt{3 * \{1 - [\cos(\delta/3)]^2\}}}{2} \quad (21c)$$

where

$$\cos\delta = 1 - 2 * \beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2 < 1 \quad (22)$$

Note that $\Delta < 0$ implies: $\beta_{\text{crest}} * C_D^2 * \Lambda_{\text{crest}}^2 < 1$.

Notation

The following symbols are used in this paper:

- A = flow cross-section area (m^2);
- C_D = discharge coefficient;
- d = flow depth (m) measured perpendicular to channel bottom;
- d_c = critical flow depth (m): i.e., flow depth at minimum specific energy;
- E = specific energy (m);
- $E_{\min}$ = minimum specific energy (m);
- g = gravity constant;
- H = depth-averaged total head (m);
- P = pressure (Pa);
- Q = water discharge (m^3/s);
- q = water discharge per unit width m^2/s;
- R = radius (m) or curvature of spillway crest;
- t = time (s);
- V = depth-averaged velocity (m/s);
- V_c = critical flow velocity (m/s): i.e., depth-averaged flow velocity at minimum specific energy;
- v = local velocity (m/s);
- W = channel width (m);
- y = distance measured perpendicular to channel bottom (m);
- z = elevation (m) taken positive upward;
- z_0 = bed elevation (m) taken positive upward;
- β = momentum correction coefficient;
- Λ = pressure correction coefficient; and
- ρ = water density (kg/m^3).

Subscript

- c = critical flow conditions (i.e., at minimum specific energy);
- crest = conditions at crest.

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