Technical Note

Momentum Considerations in Hydraulic Jumps and Bores

Hubert Chanson[1]

Abstract: A hydraulic jump is the turbulent transition from a high velocity into a slower flow. A related process is the hydraulic jump in translation. The application of the equations of conservation of mass and momentum in their integral form yields a series of relationships between the flow properties in front of and behind the jump. The effects of the cross-sectional shape and bed friction are investigated. The effect of the flow resistance yields a smaller ratio of conjugate cross-section areas for a given Froude number. The solutions are tested with some field measurements of tidal bores in natural channels, illustrating the range of cross-sectional properties in natural systems and irregular channels. **DOI: 10.1061/(ASCE)IR.1943-4774.0000409.**

CE Database subject headings: Hydraulic jump; Channels; Flow resistance; Velocity.

Author keywords: Hydraulic jumps; Bores; Momentum considerations; Irregular channel; Conjugate crosssection areas; Flow resistance.

Introduction

A hydraulic jump is the sudden transition from a high velocity flow into a slower motion. A related process is the tidal bore and positive surge, also called hydraulic jump in translation (Fig. 1). In all cases, the flow is characterized by a sudden rise in free-surface elevation and a discontinuity of the pressure and velocity fields. In the system of reference following the jump front, the integral form of the equations of conservation of mass and momentum gives a series of relationships between the flow properties in front of and behind the bore (Rayleigh 1914; Henderson 1966; Chow 1973; Liggett 1994):

$$(V_1 + U)A_1 = (V_2 + U)A_2 \quad (1)$$

$$\rho(V_1 + U)A_1[\beta_1(V_1 + U) - \beta_2(V_2 + U)] = \int_{A_2} PdA - \int_{A_1} PdA + F_{\text{fric}} - W \sin\theta \quad (2)$$

where V = flow velocity; U = bore celerity for an observer standing on the bank (Fig. 1); ρ = water density; g = gravity acceleration; A = channel cross-sectional area measured perpendicular to the main flow direction; β = momentum correction coefficient; P = pressure; the subscript 1 = the initial flow conditions; the subscript 2 = the flow conditions immediately after the jump; F_{fric} = flow resistance force; W = weight force; and θ = angle between the bed slope and horizontal. Eqs. (1) and (2) are valid for the stationary jumps ($U = 0$), tidal bores ($U > 0$), and positive surges traveling downstream ($U < 0$).

For a rectangular horizontal channel in absence of friction, a classical result is the Bélanger equation (Bélanger 1841; Chanson 2009)

$$\frac{d_2}{d_1} = \frac{1}{2}\left(\sqrt{1 + 8\mathsf{F}_1^2} - 1\right) \quad (3)$$

where d = flow depth; F_1 = upstream Froude number defined as $V_1/\sqrt{gd_1}$ for a steady jump and $(V_1 + U)/\sqrt{gd_1}$ for a bore. In open channel hydraulics, d_2 and d_1 are called the conjugate depths. Following Rayleigh (1914) and Lamb (1932), Lighthill (1978) expanded the original development of Bélanger (1841) for a non-rectangular channel, assuming implicitly small variations of the free-surface width.

In this study, the application of the momentum principle in its integral form is revisited for a hydraulic jump in an irregular channel including natural systems. The effects of the cross-sectional shape and bed friction are developed. The application to tidal bore propagation in wide shallow-water bays is discussed on the basis of recent detailed field observations.

Basic Solutions

Neglecting the flow resistance ($F_{\text{fric}} = 0$), the effect of the velocity distribution ($\beta_1 = \beta_2 = 1$) and for a flat horizontal prismatic channel, the momentum principle [Eq. (2)] becomes

$$\rho(V_1 + U)A_1(V_1 - V_2) = \int_{A_2} PdA - \int_{A_1} PdA \quad (4)$$

The difference in pressure forces may be derived assuming a hydrostatic pressure distribution in front of and behind the hydraulic jump. The net pressure force resultant consists of the increase of pressure $\rho g(d_2 - d_1)$ applied to the initial flow cross section A_1 plus the pressure force on the area $\Delta A = A_2 - A_1$:

$$\int_{A_1}^{A_2}\int \rho g(d_2 - y)dA = \frac{1}{2}\rho g(d_2 - d_1)^2 B' \quad (5)$$

where y = distance normal to the bed; d_1 and d_2 = initial and new flow depths (Fig. 1); and B' = a characteristic free-surface width. Note that $B_1 < B' < B_2$ where B_1 and B_2 are the upstream and downstream free-surface width (Fig. 1). Another characteristic free-surface width B is defined as

$$\int_{A_1}^{A_2}\int dA = A_2 - A_1 = (d_2 - d_1)B \quad (6)$$

[1]School of Civil Engineering, Univ. of Queensland, Brisbane QLD 4072, Australia. E-mail: h.chanson@uq.edu.au

Note. This manuscript was submitted on January 31, 2011; approved on August 17, 2011; published online on August 19, 2011. Discussion period open until September 1, 2012; separate discussions must be submitted for individual papers. This technical note is part of the ***Journal of Irrigation and Drainage Engineering***, Vol. 138, No. 4, April 1, 2012. ©ASCE, ISSN 0733-9437/2012/4-382–385/$25.00.

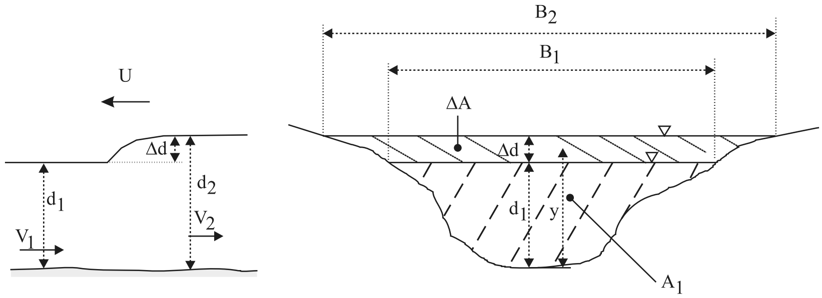

Fig. 1. Definition sketch of a hydraulic jump in a natural channel

Because the continuity equation may be rewritten

$$(V_1 - V_2) = (V_1 + U)\frac{A_2 - A_1}{A_2} \tag{7}$$

the combination of the continuity and momentum principle gives a series of relationships between the flow properties in front of and behind the jump:

$$(U + V_1)^2 = \frac{1}{2}\frac{gA_2}{A_1 B}\left[\left(2 - \frac{B'}{B}\right)A_1 + \frac{B'}{B}A_2\right] \tag{8}$$

$$(V_1 - V_2)^2 = \frac{1}{2}\frac{g(A_2 - A_1)^2}{BA_1A_2}\left[\left(2 - \frac{B'}{B}\right)A_1 + \frac{B'}{B}A_2\right] \tag{9}$$

Eq. (8) may be expressed in dimensionless terms:

$$\mathsf{F}_1^2 = \frac{(U + V_1)^2}{g\frac{A_1}{B_1}} = \frac{1}{2}\frac{A_2}{A_1}\frac{B_1}{B}\left[\left(2 - \frac{B'}{B}\right) + \frac{B'}{B}\frac{A_2}{A_1}\right] \tag{10}$$

Eq. (10) gives an analytical solution of the square of the Froude number (F_1^2) as a function of the cross-sectional ratio A_2/A_1, the ratio B'/B, and the ratio B_1/B. The Froude number definition for an irregular channel $\mathsf{F}_1 = (V_1 + U)/\sqrt{gA_1/B_1}$ is identical to the expression derived from energy considerations (Henderson 1966; Chanson 2004), but Eq. (10) is based on momentum considerations. The effects of the celerity (U) are linked with the initial flow conditions, i.e., from nil for a stationary hydraulic jump ($U = 0$) to the full extent for a fluid initially at rest ($V_1 = 0$).

Eq. (10) may be rewritten in the form of the ratio of conjugate cross-section areas A_2/A_1 as a function of the upstream Froude number F_1:

$$\frac{A_2}{A_1} = \frac{1}{2}\frac{\sqrt{\left(2 - \frac{B'}{B}\right)^2 + 8\frac{B'/B}{B_1/B}\mathsf{F}_1^2} - \left(2 - \frac{B'}{B}\right)}{\frac{B'}{B}} \tag{11}$$

that is valid for any hydraulic jump in an irregular channel. The effects of the channel cross-sectional shape are accounted for with the ratios B'/B and B_1/B.

Particular Case $B \approx B_0 \approx B_1$

In some particular situations, the approximation $B \approx B' \approx B_1$ may hold. Such cases include a rectangular channel or a channel cross-sectional shape with parallel walls next to the waterline. For $B \approx B' \approx B_1$, the combination of the continuity and momentum principles may be simplified into

$$(U + V_1)^2 = \frac{1}{2}\frac{g}{A_1}\frac{(A_1 + A_2)A_2}{B} \tag{12}$$

$$(V_1 - V_2)^2 = \frac{1}{2}\frac{g(A_1 + A_2)(A_2 - A_1)^2}{BA_1A_2} \tag{13}$$

The solution [Eqs. (11) and (13)] is a mere rewriting of the development of Lighthill (1978). Eq. (12) may be expressed in a dimensionless form as

$$\frac{A_2}{A_1} = \frac{1}{2}\left(\sqrt{1 + 8\,\mathsf{F}_1^2} - 1\right) \tag{14}$$

Eq. (14) has the same form as Eq. (3), and it yields to the Bélanger equation [Eq. (3)] for a rectangular horizontal channel in absence of friction.

Effets of Flow Resistance

In presence of some flow resistance, the momentum principle for a flat horizontal channel may be transformed. The combination of the continuity and momentum principle gives then

$$(U + V_1)^2 = \frac{1}{2}\frac{gA_2}{A_1 B}\left[\left(2 - \frac{B'}{B}\right)A_1 + \frac{B'}{B}A_2\right] + \frac{A_2}{A_2 - A_1}\frac{F_{\text{fric}}}{\rho A_1} \tag{15}$$

$$(V_1 - V_2)^2 = \frac{1}{2}\frac{g(A_2 - A_1)^2}{BA_1A_2}\left[\left(2 - \frac{B'}{B}\right)A_1 + \frac{B'}{B}A_2\right] + \frac{A_2}{A_2 - A_1}\frac{F_{\text{fric}}}{\rho g\frac{A_1^2}{B}} \tag{16}$$

In dimensionless terms, Eq. (15) may be transformed into

$$\mathsf{F}_1^2 = \frac{1}{2}\frac{A_2}{A_1}\frac{B_1}{B}\left[\left(2 - \frac{B'}{B}\right) + \frac{B'}{B}\frac{A_2}{A_1}\right] + \frac{A_2}{A_2 - A_1}\frac{F_{\text{fric}}}{\rho g\frac{A_1^2}{B}} \tag{17}$$

Eq. (17) expresses the relationship between the upstream Froude number and the ratio of the conjugate cross-section areas A_2/A_1 taking into account the flow resistance force and irregular cross-sectional shape.

Application

A number of prototype observations were carefully documented including with detailed bathymetric conditions. The reanalyzed data are summarized in Table 1. One location (Dee River) was an artificial channel section, whereas all others were natural systems. Fig. 2 shows the Sélune River channel during one field study, illustrating the wide, irregular channel cross section. The data indicated that the approximation $B \approx B' \approx B_1$ held for the Dee River but not for the other irregular channels including the Sélune River

Table 1. Field Measurements of Tidal Bores: Cross-Sectional and Hydrodynamic Properties

Reference (1)	River (2)	Date (3)	Bore type (4)	Fr_1 (5)	U m/s (6)	V_1 m/s (7)	d_1 m (8)	A_1 m² (9)	B_1 m (10)	Δd m (11)	ΔA m² (12)	B_2 m (13)	B m (14)	B' m (15)	A_1/B_1 (16)	B_2/B_1 (17)	B/B_1 (18)	B'/B_1 (19)	A_2/A_1 (20)	Fr_1 Eq. (11) (21)	Fr_1 Eq. (14) (22)
Wolanski et al. (2004)	Daly River	2/07/03	Undular	1.04	4.70	0.15	1.50	289.3	129.2	0.28	36.4	130.9	130.1	129.3	2.24	1.013	1.007	1.001	1.13	1.09	1.09
Simpson et al. (2004)	Dee River	6/09/03	Breaking	1.79	4.1	0.15	0.72	39.3	68.3	0.45	31.4	72.8	70.4	74.1	0.58	1.066	1.030	1.085	1.80	1.58	1.56
Chanson et al. (2011)	Garonne	10/09/10	Undular	1.30	4.49	0.33	1.77	105.7	75.4	0.50	39.4	81.6	78.5	76.7	1.40	1.083	1.042	1.018	1.37	1.25	1.25
	River	11/09/10	Undular	1.20	4.20	0.30	1.81	108.8	75.8	0.46	36.0	81.6	78.2	77.5	1.43	1.076	1.032	1.021	1.33	1.23	1.23
Mouazé et al. (2010)	Sélune	24/09/10	Breaking	2.35	2.00	0.86	0.38	5.25	34.7	0.34	27.3	*116.9*	80.9	66.6	0.15	3.37	2.33	1.92	6.19	2.89	3.09
	River	25/09/10	Breaking	2.48	1.96	0.59	0.33	3.56	33.2	0.41	31.3	*117.0*	77.3	65.7	0.11	3.53	2.33	1.98	9.79	4.46	4.76

Note: d_1: initial water depth at sampling location; italic data: incomplete data.

Fig. 2. Tidal bore of the Sélune River (France) on 24 September 2010; bore propagation from the right to the left; in the background the bore expanding over the sand shoals

channel (Table 1). In these irregular channels, the data yielded consistently

$$B_1 < B' < B < B_2 \tag{18}$$

as shown in Table 1 (columns 9, 12, 13, and 14). The upstream Froude number was estimated from the velocity measurements (column 5) and the data are summarized in Fig. 3. Fig. 3 presents the upstream Froude number as a function of the ratio of conjugate cross-section areas. The data are compared with Eqs. (11) and (14) and the solution of the Bélanger equation [Eq. (3)]. The results highlighted, first, the effects of the irregular cross section. The Bélanger equation based on the assumption of a rectangular channel is inappropriate in an irregular channel, as illustrated by the difference between Eqs. (3) and (11). The effects of the irregular channel cross section increase with increasing Froude number and bore height Δd. Second, the field data were predicted reasonably well by Eq. (11) except for the last data point (Sélune River,

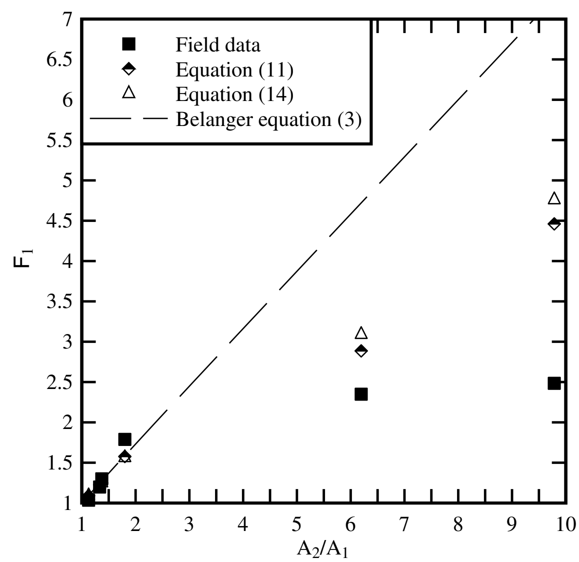

Fig. 3. Relationship between Froude number and ratio of conjugate cross-section areas; comparison between field observations and the solutions of Eqs. (3), (11), and (14)

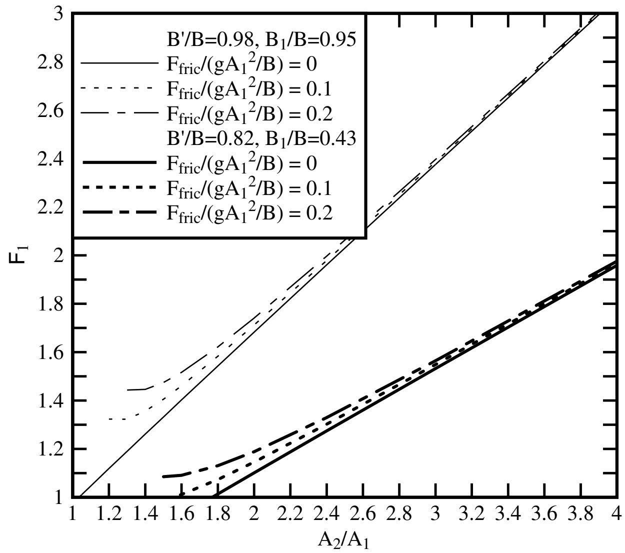

Fig. 4. Effects of the flow resistance on the solution of the momentum equation [Eq. (17)] for $B'/B = 0.98$ and $B_1/B = 0.95$ (Garonne River), and (B) $B'/B = 0.82$ and $B_1/B = 0.43$ (Sélune River)

25 Sept. 2010). Third, the upstream Froude number definition $\mathsf{F}_1 = (V_1 + U)/\sqrt{gA_1/B_1}$ may differ significantly to the traditional approximation $(V_1 + U)/\sqrt{gd_1}$. For the field data in natural irregular channels (Table 1), the differences ranged from 12 to 74%.

Effects of Bed Friction

The effects of bed friction on the hydraulic jump properties were tested on irregular channels. Fig. 4 presents the upstream Froude number as a function of ratio of the conjugate crosssection areas A_2/A_1 for values of B'/B and B_1/B corresponding to the bathymetric conditions of the Garonne River and Sélune River (Table 1).

For a given Froude number, the theoretical considerations imply a smaller ratio of the conjugate cross-section areas A_2/A_1 hence, a smaller ratio of conjugate depths d_2/d_1, with increasing flow resistance to satisfy momentum considerations (Fig. 4). Although the finding is intuitive and consistent with physical data in rectangular channels (Leutheusser and Schiller 1975; Pagliara et al. 2008), Eq. (17) is general and applies to any cross-sectional shape. However, the effects of flow resistance decrease with increasing Froude number, becoming small for upstream Froude numbers greater than 2 to 3 depending on the cross-sectional properties (Fig. 4).

Conclusion

The application of the equations of conservation of mass and momentum in their integral form is revisited for a hydraulic jump in an irregular channel. Some complete solutions are developed expressing the ratio of the conjugate cross-section areas as a function of the upstream Froude number $\mathsf{F}_1 = (V_1 + U)/\sqrt{gA_1/B_1}$ for a range of channel cross sections. The effects of the flow resistance are observed to decrease the ratio of conjugate depths for a given Froude number. The solutions were tested with some field measurements in natural irregular channels. The results illustrate that the Bélanger equation is not applicable and that the cross-sectional properties of irregular channels have a significant impact on the flow properties.

Notation

The following symbols are used in this paper:

- A = flow cross-section area (m^2);
- B = (1) free-surface width (m); (2) characteristic free-surface width (m) [Eq. (7)];
- B' = characteristic free-surface width (m) [Eq. (5)];
- d = flow depth;
- F_{fric} = flow resistance (N);
- F = Froude number: for an irregular channel: $\mathsf{F} = (V + U)/\sqrt{gA/B}$;
- g = gravity acceleration (m/s^2);
- P = pressure (Pa);
- U = bore celerity (m/s) positive upstream;
- V = flow velocity (m/s) positive downstream;
- W = weight force (N);
- y = vertical elevation (m) above the bed;
- θ = angle between bed slope and horizontal, positive downwards; and
- ρ = water density (kg/m^3).

Subscripts

- 1 = upstream or initial flow conditions; and
- 2 = downstream or new flow conditions.

References

Bélanger, J. B. (1841). "Notes sur l'hydraulique." *Notes on hydraulic engineering*, Ecole Royale des Ponts et Chaussées, Paris (in French).

Chanson, H. (2004). *The hydraulics of open channel flow: An introduction*, 2nd Ed., Butterworth-Heinemann, Oxford, UK.

Chanson, H. (2009). "Development of the Bélanger equation and backwater equation by Jean-Baptiste Bélanger (1828)." *J. Hydraul. Eng.*, 135(3), 159–163.

Chanson, H., Reungoat, D., Simon, B., and Lubin, P. (2011). "High-frequency turbulence and suspended sediment concentration measurements in the Garonne River tidal bore." *Estuarine Coastal Shelf Sci.*, 95(2–3), 298–306.

Chow, V. T. (1973). *Open channel hydraulics*, McGraw-Hill International, New York.

Henderson, F. M. (1966). *Open channel flow*, MacMillan, New York.

Leutheusser, H. J., and Schiller, E. J. (1975). "Hydraulic jump in a rough channel." *Water Power Dam Constr.*, 27(5), 186–191.

Lamb, H. (1932). *Hydrodynamics*, 6th Ed., Cambridge University Press, Cambridge, UK.

Liggett, J. A. (1994). *Fluid mechanics*, McGraw-Hill, New York.

Lighthill, J. (1978). *Waves in fluids*, Cambridge University Press, Cambridge, UK.

Mouazé, D., Chanson, H., and Simon, B. (2010). "Field measurements in the tidal bore of the Sélune River in the Bay of Mont Saint Michel (September 2010)." *Hydraulic Model Rep. No. CH81/10*, School of Civil Engineering, Univ. of Queensland, Brisbane, Australia.

Pagliara, S., Lotti, I., and Palermo, M. (2008). "Hydraulic jump on rough bed of stream rehabilitation structures." *J. Hydro-Environ. Res.*, 2(1), 29–38.

Rayleigh, Lord (1914). "On the theory of long waves and bores." *Proc. R. Soc. London, Ser. A*, 90(619), 324–328.

Simpson, J. H., Fisher, N. R., and Wiles, P. (2004). "Reynolds stress and TKE production in an estuary with a tidal bore." *Estuarine Coastal Shelf Sci.*, 60(4), 619–627.

Wolanski, E., Williams, D., Spagnol, S., and Chanson, H. (2004). "Undular tidal bore dynamics in the Daly Estuary, Northern Australia." *Estuarine Coastal Shelf Sci.*, 60(4), 629–636.